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Motion of a solid sphere in a general flow near a plane boundary at zero Reynolds number

R.G. COX*
Department of Civil Engineering and Applied Mechanics, McGill University, Montreal, Quebec, Canada H3A 2K6

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Abstract. It is demonstrated how the hydrodynamic force and moment of force acting on a solid sphere may be calculated when it is placed at rest at an arbitrary position in a two dimensional flow at zero Reynolds number in which the region of flow is bounded by either an undeformable planar free surface or by a plane solid wall. The results so obtained are used to calculate the motion of a freely moving solid sphere in an asymmetric vortex in the presence of an underformable free surface. It is seen that the sphere, depending on the direction of the undisturbed flow, will either spiral into or out of the vortex. This implies that when a dilute suspension of such spherical particles undergoes such a vortex motion in the presence of the free surface, the vortex will either fill up with particles from the surrounding flow or become devoid of particles.

1. Introduction

Consider a solid sphere of a radius a translating with velocity $\mathbf{V}$ and rotating with angular velocity $\boldsymbol{\Omega}$ in an unbounded fluid of viscosity μ undergoing a prescribed undisturbed fluid flow (with velocity $\mathbf{U}(\mathbf{r})$ at position $\mathbf{r}$) for which the Reynolds number is zero so that inertia effects on the fluid flow are negligible. The hydrodynamic force $\mathbf{F}$ and moment of force $\mathbf{G}$ (about the sphere's centre) acting on the sphere are then given by Faxén's laws [1,2] as

$$\mathbf{F} = 6\pi\mu a\left[\mathbf{U}|_C - \mathbf{V}\right] + \mu\pi a^3\nabla^2\mathbf{U}|_C, \tag{1.1a}$$

$$\mathbf{G} = 8\pi\mu a^3\left[\frac{1}{2}(\nabla\times\mathbf{U})|_C - \boldsymbol{\Omega}\right], \tag{1.1b}$$

where $|C$ denotes evaluation at the point occupied by the sphere's centre.

Completely general results like (1.1) are, however, not known for flows which are bounded by solid walls or free surfaces. Instead, if the sphere's radius a is very much smaller than the distance (L say) of the sphere from any of the boundaries, the method of reflections may be used to obtain a solution as an expansion in the small parameter a/L. This has been done for many situations in which the fluid is bounded by solid surfaces, with various authors considering, for example, boundaries in the form of a single plane wall, a pair of parallel plane walls, or an infinitely long circular cylinder (see Happel and Brenner [3], Chapter 7).

When the distance L of the sphere to the boundaries is of the same order of magnitude as the sphere's radius a, then relatively fewer problems have been solved. These involve rather simple boundaries (such as a plane solid wall or a concentric solid spherical wall) with the undisturbed fluid velocity $\mathbf{U}(\mathbf{r})$ taking on a particularly simple form. Thus O'Neill [4], Goldman *et al.* [5] and Brenner [6], using spherical bipolar coordinates, obtained solutions for

* Deceased, July 31, 1995

a purely translating solid sphere ($\boldsymbol{\Omega} = \mathbf{0}$) in a quiescent fluid ($\mathbf{U} = \mathbf{0}$) bounded by a solid plane wall for arbitrary distances between the sphere and the wall. Also, using the same method, solutions for a purely rotating solid sphere ($\mathbf{V} = \mathbf{0}$) in a quiescent fluid ($\mathbf{U} = \mathbf{0}$) bounded by a plane solid wall have been obtained by Jeffery [7], and Dean & O'Neill [8], whilst Goldman *et al.* [9] obtained the solution for a sphere at rest ($\mathbf{V} = \boldsymbol{\Omega} = \mathbf{0}$) in a planar shear flow bounded by a solid wall. Results for similar problems involving a solid sphere moving in a fluid bounded by an undeformable free surface may be obtained, by making use of symmetry, from the two-sphere problems considered by Goldman *et al.* [10] (see also §4.2 of present paper).

The above results are extended in the present paper in which the hydrodynamic force and moment of force on a solid sphere placed at rest at a general position in a fluid flow are obtained for zero Reynolds number when the fluid is bounded by either an undeformable planar free surface or a planar solid surface and is undergoing a prescribed two dimensional undisturbed flow of general polynomial form (with degree ≤ 10).Thus, in §2, the general theory is given for obtaining the force and moment of force on the sphere whilst in §3 all the various polynomial two dimensional undisturbed flows are listed for both flows bounded by an undeformable free surface and bounded by a solid surface. Then, in §4, the general theory given in §2 is applied to all the various undisturbed flows listed in §3.

The results obtained in §4 are then applied in §5 to a particular example in which the fluid, bounded by an undeformable free surface, undergoes a two dimensional undisturbed flow which is in the form of a bounded vortex neighbouring the surface. The motion of a freely moving sphere (i.e. one for which there is no external force or moment of force about its centre acting on it) in the vortex flow is thus obtained. It is shown that the sphere moves into the vortex and spirals towards the vortex centre, ending up close to the zero velocity point of the vortex. In this process the path of the sphere centre is not along a streamline but migrates steadily from one streamline to another. It is also noted in §6 that by considering the same undisturbed flow, but with the velocity reversed everywhere, a sphere initially in the vortex spirals outwards and finally leaves the vortex. This means that for a dilute suspension of such spherical particles undergoing such flows, the spheres either concentrate in or leave the vortex (depending on the direction of the undisturbed flow).

This behaviour of particles migrating across streamlines and particles in a dilute suspension tending to become more (or less) concentrated in certain regions of the flow is well known to occur at non-zero Reynolds number as a result of the effects of fluid inertia. However, it is not so well known that similar effects can occur at zero Reynolds number purely as a result of the effect of boundaries. At non-zero Reynolds number, for example, a freely moving small sphere in Poiseuille flow along either a circular cylinder or between parallel solid walls will move across streamlines to a position approximately half way between the centre and the wall [11, 12, 13, 14, 15, 16], resulting in particles in a dilute suspension becoming more concentrated at such a position, a phenomenon known as the 'tubular pinch' effect. However, at zero Reynolds number, while similar effects do not occur for the relatively simple flows previously studied (such as Poiseuille flow or shear flow), they can occur for more complicated flows with boundaries present (such as the vortex flow considered here in §5). This phenomenon of particles moving across streamlines and, in a suspension, tending to concentrate (or move out of) certain regions of the flow is, at zero Reynolds number, an effect of the boundaries present. If fact, it is shown in §6 that for an unbounded flow (for which Faxén's laws (1.1) apply), a dilute suspension of particles has no tendency for the concentration to increase (or decrease) in any region of the flow, although locally individual particles can move across streamlines.

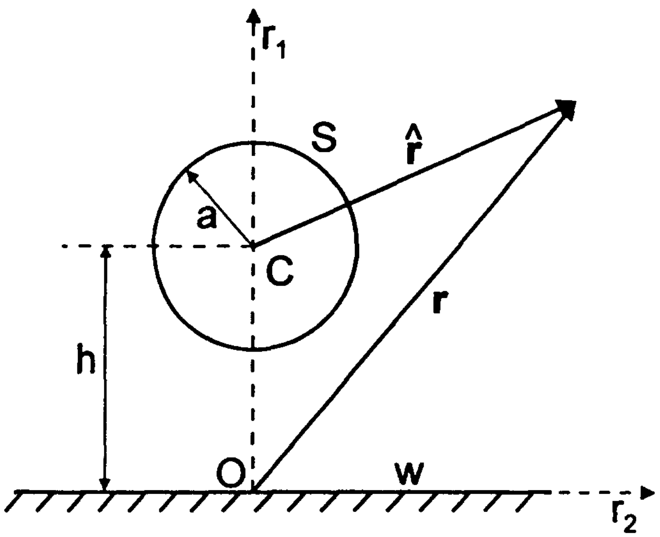


Fig. 1. Solid sphere placed in a flowing fluid occupying the half-space $r_1 > 0$ and bounded by either an undeformable free surface or a solid surface at $r_1 = 0$.

2. Force and torque on a solid sphere

Consider a fluid of viscosity μ occupying the half-space $r_1 \geq 0$ with (r_1, r_2, r_3) being a fixed set of Cartesian coordinates. This fluid is assumed to be undergoing a prescribed flow at zero Reynolds number with velocity **U**, pressure P and stress tensor Σ_{ij} so that for $r_1 \geq 0$

$$\Sigma_{ij,j} = 0, \tag{2.1}$$

$$U_{i,i} = 0, \tag{2.2}$$

where the stress tensor Σ_{ij} is

$$\Sigma_{ij} = -P\delta_{ij} + \mu(U_{i,j} + U_{j,i}). \tag{2.3}$$

Into this flow a solid sphere of radius a is placed at rest (with no rotation or translation) with its centre at position $(h, 0, 0)$ where $h > a$ [see Fig. 1]. We will demonstrate in this section how the force and torque on this sphere may be calculated. With the sphere present we take the velocity to be **u**, pressure to be p, and stress tensor to be σ_{ij}, so that the disturbance velocity and pressure produced by the presence of the sphere is (**u** – **U**) and $(p - P)$ respectively. Then, like the undisturbed flow (**U**, P), the flow (**u**, p) with the sphere present must satisfy

$$\sigma_{ij,j} = 0, \tag{2.4}$$

$$u_{i,i} = 0, \tag{2.5}$$

where

$$\sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{2.6}$$

The no slip boundary condition then requires that on the sphere surface S

$$u_i = 0, \tag{2.7}$$

whilst, since the disturbance flow must tend to zero at infinity,

$$u_i \sim U_i \quad \text{as } r \to \infty, \tag{2.8}$$

where $r = \sqrt{r_j r_j}$ is the distance from the origin O of coordinates. For the planar boundary $r_1 = 0$ of the fluid region we will consider two situations, the first being that in which $r_1 = 0$ is an undeformable free surface (discussed in sub-section 2.1) and the second being that in which the fluid is bounded by a stationary plane solid wall at $r_1 = 0$ (discussed in sub-section 2.2). The former situation of an undeformable free surface would occur with a liquid bounded by an interface (at $r_1 = 0$) with infinitely large surface tension, there being either no fluid or a fluid of negligible viscosity in the region $r_1 < 0$. At such a boundary one would have zero normal velocity and zero tangential stress, so that for the flow $(\mathbf{U}, P)$ one must have

$$U_1 = 0 \quad \Sigma_{21} = \Sigma_{31} = 0 \quad \text{on } r_1 = 0 \tag{2.9}$$

and for the flow $(\mathbf{u}, p)$

$$u_1 = 0 \quad \sigma_{21} = \sigma_{31} = 0 \quad \text{on } r_1 = 0. \tag{2.10}$$

For the situation in which the fluid is bounded by a stationary plane solid wall at $r_1 = 0$, the fluid velocity must be zero there so that for the flow $(\mathbf{U}, P)$ one must have

$$U_1 = U_2 = U_3 = 0 \quad \text{on } r_1 = 0 \tag{2.11}$$

and for the flow $(\mathbf{u}, p)$

$$u_1 = u_2 = u_3 = 0 \quad \text{on } r_1 = 0. \tag{2.12}$$

2.1. Free surface at $r_1 = 0$

Consider a fluid bounded by an undeformable free surface at $r_1 = 0$ so that the undisturbed flow $(\mathbf{U}, P)$ satisfies (2.1)–(2.3) in the region $r_1 > 0$ with boundary conditions (2.9) on $r_1 = 0$ whilst the flow $(\mathbf{u}, p)$ with the sphere present satisfies (2.4)–(2.6) in the region $r_1 > 0$ with boundary conditions (2.10) on $r_1 = 0$.

We define ${}^F u_{Tik}$ as the ith component of the zero Reynolds number velocity field produced by the sphere (of radius a with center at $(h, 0, 0)$) translating in the k-direction with unit velocity in an otherwise quiescent fluid bounded by the undeformable free surface at $r_1 = 0$. The corresponding pressure field is defined as ${}^F p_{Tk}$ whilst the ij-component of the corresponding stress tensor is defined as ${}^F \sigma_{Tijk}$. Then in the fluid region we have the flow $({}^F \mathbf{u}_{Tk}, {}^F p_{Tk}$ satisfying

$${}^F \sigma_{Tijk,j} = 0, \tag{2.13}$$

$${}^F u_{Tik,i} = 0, \tag{2.14}$$

where

$${}^F \sigma_{Tijk} = -{}^F p_{Tk} \delta_{ij} + \mu({}^F u_{Tik,j} + {}^F u_{Tjk,i}), \tag{2.15}$$

with the boundary condition on the sphere surface S of

$${}^F u_{Tik} = \delta_{ik}, \tag{2.16}$$

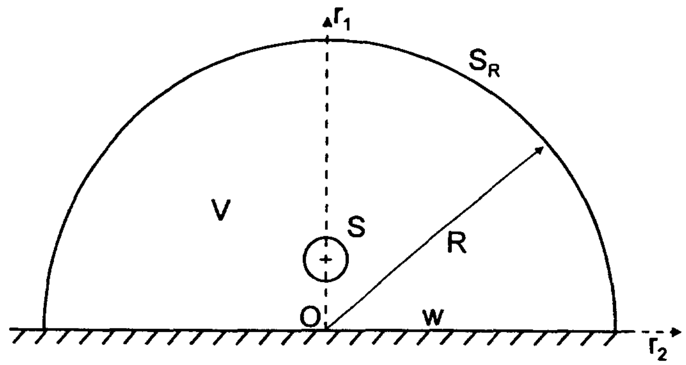


Fig. 2. Definition of the fluid volume V bounded by the hemispherical surface S_R of radius R, the boundary W at $r_1 = 0$ and the surface S of the sphere.

with δ_{ik} being the Kronecker delta), the boundary condition on the undeformable free surface W at $r_1 = 0$ of

$$^F u_{T1k} = 0, \quad ^F\sigma_{T21k} = {}^F\sigma_{T31k} = 0, \tag{2.17}$$

whilst at large distances we must have

$$^F u_{Tik} \to 0 \qquad \text{as } r \to \infty. \tag{2.18}$$

Since (u_i, p) and $(^F u_{Tik}, {}^F p_{Tk})$ are both zero Reynolds number (creeping) flows within the region $r_1 \geq 0$ exterior to the sphere surface S, we apply the Lorentz [17] reciprocal theorem to these flows for the fluid volume V contained within a large sphere, S_R of radius $R(R >> h)$ with centre at the origin O [see Fig. 2]. We thus obtain

$$\int_{S+S_R+W} (u_i {}^F\sigma_{Tijk} - {}^F u_{Tik}\sigma_{ij}) n_j \, \mathrm{d}S = 0, \tag{2.19}$$

where n_j is the unit normal vector directed out of V and dS is an element of surface area. This surface integral in (2.19) is taken over the surfaces S, S_R and the part of W that form the boundary of the volume V. From the boundary conditions (2.10) and (2.17), it is seen that the integral over W in (2.19) is identically zero whilst from the boundary conditions (2.7) and (2.16) it is seen that the integral over the sphere surface S is equal to the force F_k on the sphere due to the flow field $(\mathbf{u}, p)$. Thus

$$F_k = \int_{S_R} (^F u_{Tik}\sigma_{ij} - u_i {}^F\sigma_{Tijk}) n_j \, \mathrm{d}S. \tag{2.20}$$

It is assumed that the given flow field (U, P) is not singular at the origin O on the boundary or anywhere else in the fluid region $r_1 \geq 0$ so that U (and P) may be expanded as a Taylor series in the coordinates r_1, r_2, r_3. A general term in this expansion for U will be homogeneous in r^n where n is some positive integer (or zero). Since the problem of solving for the force and torque on the solid sphere placed in this flow is linear, we need only consider an undisturbed flow in which U is proportional to r^n (where n is a positive integer) so that the corresponding pressure P and stress tensor Σ_{ij} are proportional to r^{n-1}. The asymptotic forms of u_i and σ_{ij} for $r \to \infty$ must then be

$$u_i = U_i(\text{ of order } r^n\ ) + (\text{ terms of order } r^{-1}, r^{-2} \ldots), \tag{2.21}$$

$$\sigma_{ij} = \Sigma_{ij}(\text{ of order } r^{n-1}) + (\text{ terms of order } r^{-2}, r^{-3} \ldots). \tag{2.22}$$

Also, the asymptotic forms of ${}^F u_{Tik}$ and ${}^F\sigma_{Tik}$ for $r \to \infty$ must likewise be

$${}^F u_{Tik} = \text{ terms of order } r^{-1}, r^{-2} \ldots \tag{2.23}$$

$${}^F\sigma_{Tijk} = \text{ terms of order } r^{-2}, r^{-3} \ldots. \tag{2.24}$$

In deriving the asymptotic forms (2.21)–(2.24) it has been noted that the disturbance velocities ($u_i - U_i$ or ${}^F u_{Tik}$) produced by the sphere as $r \to \infty$ be the flows produced by a point force, a point force doublet, etc. and so contain terms of order $r^{-1}, r^{-2} \ldots$.

By substituting the asymptotic forms (2.21)–(2.24) for $r \to \infty$ into the integral in (2.20), we see that by equating terms of order R^0 in (2.20) in the limit $R \to \infty$, we obtain the hydrodynamic force $\mathbf{F}$ on the sphere as

$$F_k = \int_{S_R} {}_{(-n-i)}u_{Tik}\Sigma_{ij} - {}_{(-n-1)}{}^F\sigma_{Tijk}U_i)n_j \, \mathrm{d}S, \tag{2.25}$$

where ${}_{(-n-1)}{}^F u_{Tik}$ is the term homogeneous in r^{-n-1} in the asymptotic expansion of ${}^F u_{Tik}$ for $r \to \infty$ and where ${}_{(-n-1)}{}^F\sigma_{Tijk}$ is the corresponding stress tensor (and hence is homogeneous in r^{-n-1} as $r \to \infty$) which is given by

$${}_{(-n-1)}{}^F\sigma_{Tijk} = -{}_{-(n-1)}{}^F p_{Tk}\delta_{ij} + \mu\left({}_{(-n-1)}{}^F u_{Tik,j} + {}_{(-n-1)}{}^F u_{Tjk,i}\right), \tag{2.26}$$

where ${}_{(-n-1)}{}^F p_{Tk}$ is the corresponding pressure, being the term homogeneous in r^{-n-2} in the asymptotic expansion of the pressure ${}^F p_{Tk}$ for the limit $r \to \infty$.

In order to determine the moment of force $\mathbf{G}$ on the stationary solid sphere S about its centre (C say) due to the flow $(\mathbf{u}, p)$ we define ${}^F u_{Rik}$ as the ith component of the velocity fluid (for zero Reynolds number flow) for the flow produced by the sphere rotating wth its centre C at rest with unit angular velocity directed in the positive k-direction in an otherwise quiescent fluid bounded by the undeformable free surface at $r_1 = 0$. If the corresponding pressure field is defined as ${}^F p_{Rk}$ and stress tensor field as ${}^F\sigma_{Rijk}$, then in the fluid region $r_1 > 0$ we have the flow (${}^F\mathbf{u}_{Rk}$, ${}^F p_{Rk}$) satisfying

$${}^F\sigma_{Rijk,j} = 0, \tag{2.27}$$

$${}^F u_{Rik,i} = 0, \tag{2.28}$$

where

$${}^F\sigma_{Rijk} = -{}^F p_{Rk}\delta_{ij} + \mu\left({}^F u_{Rik,j} + {}^F u_{Rjk,i}\right) \tag{2.29}$$

with boundary condition on the sphere surface S of

$${}^F u_{Rik} = \epsilon_{ikj}\hat{\mathbf{r}}_j, \tag{2.30}$$

where ϵ_{ikj} is the alternating tensor and $\hat{\mathbf{r}}$ is the position vector relative to the centre C of the sphere (see Fig. 1). On the undeformable free surface W at $r_1 = 0$, the boundary condition is

$${}^F u_{R1k} = 0 \qquad {}^F\sigma_{R21k} = {}^F\sigma_{31k} = 0, \tag{2.31}$$

whilst at large distances,

$$ {}^{F}u_{Rik} \to 0 \qquad \text{as } r \to \infty. \tag{2.32}$$

By applying the Lorentz [17] reciprocal theorem to the flows (u_i, p) and $({}^{F}u_{Rik}, {}^{F}p_{Rk})$ we obtain in a manner similar to that for (2.19)

$$\int_{S+S_R+W} \left(u_i {}^{F}\sigma_{Rik} - {}^{F}u_{Rik}\sigma_{ij}\right) n_j \,\mathrm{d}S = 0. \tag{2.33}$$

Again the integral over W is identically zero whilst that over the sphere surface S is, by (2.7) and (2.30), equal to the moment of force G_k on the sphere about its centre C due to the flow field $(\mathbf{u}, p)$.Thus

$$G_k = \int_{S_R} \left({}^{F}u_{Rik}\sigma_{ij} - u_i {}^{F}\sigma_{Rijk}\right) n_j \,\mathrm{d}S. \tag{2.34}$$

Proceeding as before, if U_i is again proportional to r^n (where n, is a positive integer), we obtain the moment of force $\mathbf{G}$ on the sphere about its centre C by equating terms of order R^0 in the asymptotic form of (2.34) for $R \to \infty$ as

$$G_k = \int_{S_R} \left({}_{(-n-1)}{}^{F}u_{Rik}\Sigma_{ij} - {}_{(-n-1)}{}^{F}\sigma_{Rijk}U_i\right) n_j \,\mathrm{d}S, \tag{2.35}$$

where ${}_{(-n-1)}{}^{F}u_{Rik}$ is the term homogeneous in r^{-n-1} in the asymptotic expansion of ${}^{F}u_{Rik}$ for $r \to \infty$ and where ${}_{(-n-1)}{}^{F}\sigma_{Rik}$ is the corresponding stress tensor (homogeneous in r^{-n-2}) given by

$${}_{(-n-1)}{}^{F}\sigma_{Rijk} = -{}_{(-n-1)}{}^{F}p_{Rk}\delta_{ij} + \mu\left({}_{(-n-1)}{}^{F}u_{Rik,j} + {}_{(-n-1)}{}^{F}u_{Rjk,i}\right) \tag{2.36}$$

with ${}_{(-n-1)}{}^{F}p_{Rk}$ being the corresponding pressure and hence is the term homogeneous in r^{-n-2} in the asymptotic expansion of the pressure ${}^{F}p_{Rk}$ for the limit $r \to \infty$.

2.2. SURFACE AT $r_1 = 0$

We consider now a fluid bounded by a solid wall at $r_1 = 0$ (i.e. at the surface W) so that the undisturbed flow $(\mathbf{U}, P)$ again satisfies (2.1)–(2.3) in the region $r_1 > 0$ but with boundary conditions (2.11) on $r_1 = 0$ whilst the flow $(\mathbf{u}, p)$ again satisfies (2.4)–(2.6) in the region $r_1 > 0$ but with boundary conditions (2.12) on $r_1 = 0$.

We proceed as in the previous sub-section and define ${}^{S}u_{Tik}$ in a manner similar to ${}^{F}u_{Tik}$ as the ith component of the velocity field (for zero Reynolds number) for the sphere translating with unit velocity in the k-direction in an otherwise quiescent fluid but bounded by a solid wall at $r_1 = 0$, so that ${}^{S}u_{Tik}$ satisfies equations like (2.13)–(2.15) with boundary conditions like (2.16) and (2.18) but with boundary condition (2.17) on $r_1 = 0$ replaced by

$${}^{S}u_{T1k} = {}^{S}u_{T2k} = {}^{S}u_{T3k} = 0. \tag{2.37}$$

Likewise we also define ${}^{S}u_{Rik}$ in a manner similar to ${}^{F}u_{Rik}$ as the ith component of the velocity field (at zero Reynolds number) for a sphere rotating without translation about its centre C with unit angular velocity directed in the positive k-direction in an otherwise quiescent fluid but bounded by a solid wall at $r_1 = 0$, so that ${}^{S}u_{Rik}$ satisfies equations like (2.27)–(2.29) with boundary conditions like (2.30) and (2.32) but with boundary condition (2.31) on $r_1 = 0$ replaced by

$$^{S}u_{R1k} = {}^{S}u_{R2k} = {}^{S}u_{R3k} = 0. \tag{2.38}$$

With $^{S}u_{Tik}$ and $^{S}u_{Rik}$ replacing $^{F}u_{Tik}$ and $^{F}u_{Rik}$ respectively in the previous sub-section we obtain the force **F** and moment of force **G** (about the sphere centre C) acting on the sphere placed at rest in the flow (**U**, P) bounded by a plane solid wall at $r_1 = 0$ as (see (2.25) and (2.35))

$$F_k = \int_{S_R} \left({}_{(-n-1)}{}^{S}u_{Til}\Sigma_{ij} - {}_{(-n-1)}{}^{S}\sigma_{Tijk}U_i \right) n_j \, \mathrm{d}S, \tag{2.39}$$

$$G_k = \int_{S_R} \left({}_{(-n-1)}{}^{S}u_{Rik}\Sigma_{ij} - {}_{(-n-1)}{}^{S}\sigma_{Rijk}U_i \right) n_j \, \mathrm{d}S, \tag{2.40}$$

where ${}_{(-n-1)}{}^{S}u_{Tik}$ and ${}_{(-n-1)}{}^{S}u_{Rik}$ are the terms homogeneous in r^{-n-1} in the asymptotic expansions of $^{S}u_{Tik}$ and $^{S}u_{Rik}$ respectively for $r \to \infty$, with ${}_{(-n-1)}{}^{S}\sigma_{Tijk}$ and ${}_{(-n-1)}{}^{S}\sigma_{Rijk}$ being the corresponding stress tensors.

3. The undisturbed flow field (U, P)

For simplicity we will consider only given undisturbed flows (**U**, P) which are two dimensional (in the r_1, r_2 plane say) with **U** $= (U_1, U_2, 0)$ in which U_1, U_2 and P are functions only of the coordinates r_1 and r_2. The velocity field **U**, since it satisfies the continuity Eq. (2.2), may then be expressed in terms of a streamfunction ψ with

$$U_1 = \frac{\partial \psi}{\partial r_2} \qquad U_2 = -\frac{\partial \psi}{\partial r_1}. \tag{3.1}$$

The Eqs. (2.1) and (2.3) for **U** then show that ψ satisfies the biharmonic equation in the r_1, r_2 plane, i.e.

$$\nabla^4 \psi = 0. \tag{3.2}$$

If we use plane polar coordinates ρ, θ in the r_1, r_2 plane as defined in Fig. 3, then (3.1) may be written as

$$U_\rho = \frac{1}{\rho}\frac{\partial \psi}{\partial \theta}, \qquad U_\theta = -\frac{\partial \psi}{\partial \rho}, \tag{3.3}$$

where U_ρ and U_θ are respectively the ρ and θ components of **U**. The biharmonic Eq. (3.2) then yields

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \psi = 0. \tag{3.4}$$

If we take ψ to be proportional to ρ^N (where N is a positive integer), the corresponding velocity **U** is then proportional to r^{N-1} so that in the previous section the integer n, which was used is $n = N - 1$. The solution of (3.4) with ψ of such a form may readily be shown to be.

$$\psi = \rho^N (a_1 \sin N\theta + a_2 \cos N\theta + a_3 \sin(N-2)\theta + a_4 \cos(N-2)\theta). \tag{3.5}$$

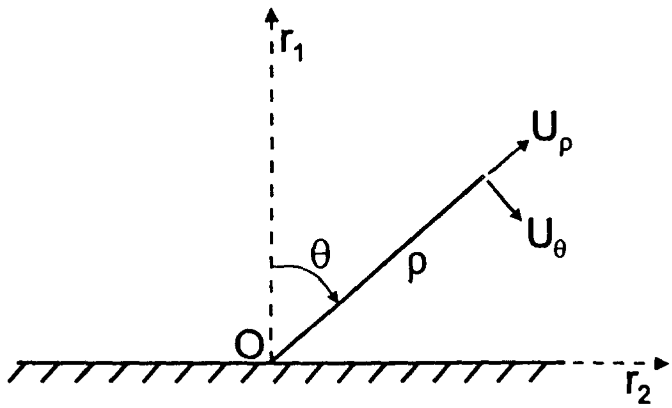


Fig. 3. Plane polar coordinates (ρ, θ) used to define the two dimensional undisturbed flow.

where a_1, a_2, a_3 and a_4 are constants. The pressure field P corresponding to this flow may readily be shown to be given by

$$\frac{P}{\mu} = 4\,(N-1)\,\rho^{N-2}\,(a_3 \cos{(N-2)\,\theta} - a_4 \sin{(N-2)\,\theta})\,. \tag{3.6}$$

In the following two subsections we consider separately the undisturbed flow field determined by (3.5) and (3.6) for the situations of an undeformed free surface and of a solid surface at $r_1 = 0$.

3.1. Free surface

For an undeformable free surface at $r_1 = 0$, the boundary conditions (2.9), when expressed in terms of the stream function ψ, reduce to

$$\psi = 0 \qquad \text{on both } \theta = -\frac{\pi}{2} \text{ and } \theta = +\frac{\pi}{2}, \tag{3.7}$$

$$\frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad \text{on both } \theta = -\frac{\pi}{2} \text{ and } \theta = +\frac{\pi}{2}. \tag{3.8}$$

Substituting the value of ψ given by (3.5) in these boundary conditions and solving, we see that the independent solutions so obtained fall into two classes; namely those for which ψ is an odd function of θ and those for which ψ is an even function of θ. In the former class, which we will refer to as being 'symmetric' flows, we have

$$\psi = \rho^N \sin N\theta \qquad \text{where } N = 2, 4, 6 \ldots \tag{3.9a}$$

$$\psi = \rho^N \sin(N-2)\theta \qquad \text{where } N = 4, 6, 8 \ldots \tag{3.9b}$$

and in the latter class, which we will refer to as 'antisymmetric' flows,

$$\psi = \rho^N \cos N\theta \qquad \text{where } N = 3, 5, 7 \ldots \tag{3.10a}$$

$$\psi = \rho^N \cos(N-2)\theta \qquad \text{where } N = 3, 5, 7 \ldots \tag{3.10b}$$

Table 1. Values of U_1, U_2, and P/μ in Cartesian coordinates for the 'symmetric' flows (3.9a, b) bounded by a free surface at $r_1 = 0$.

ψ	U_1	U_2	P/μ
$\rho^2 \sin 2\theta$	$2r_1$	$-2r_2$	0
$\rho^4 \sin 4\theta$	$4r_1(r_1^2 - 3r_2^2)$	$4r_2(-3r_1^2 + r_2^2)$	0
$\rho^4 \sin 2\theta$	$2r_1(r_1^2 + 3r_2^2)$	$2r_2(-3r_1^2 - r_2^2)$	$12(r_1^2 - r_2^2)$
$\rho^6 \sin 6\theta$	$6r_1(r_1^4 - 10r_1^2r_2^2 + 5r_2^4)$	$6r_2(-5r_1^4 + 10r_1^2r_2^2 - r_2^4)$	0
$\rho^6 \sin 4\theta$	$4r_1(r_1^4 - 5r_2^4)$	$4r_2(-5r_1^4 + r_2^4)$	$20(r_1^4 - 6r_1^2r_2^2 + r_2^4)$

Table 2. Values of U_1, U_2, and P/μ in Cartesian coordinates for the 'antisymmetric' flows (3.10a, b) bounded by a free surface at $r_1 = 0$.

ψ	U_1	U_2	P/μ
$\rho^3 \cos 3\theta$	$-6r_1r_2$	$3(-r_1^2 + r_2^2)$	0
$\rho^3 \cos \theta$	$2r_1r_2$	$-3r_1^2 - r_2^2$	$-8r_2$
$\rho^3 \cos 5\theta$	$20r_1r_2(-r_1^2 + r_2^2)$	$5(-r_1^4 + 6r_1^2r_2^2 - r_2^4)$	0
$\rho^3 \cos 3\theta$	$4r_1r_2(-r_1^2 - 3r_2^2)$	$-5r_1^4 + 6r_1^2r_2^2 + 3r_2^4$	$16r_2(-3r_1^2 + r_2^2)$

the uniform flow given by $\psi = \rho \cos\theta$ having been omitted (since it is the same problem as that of the translating sphere in a quiescent fluid).

When the solid sphere is placed into each of these flows, it is observed that, by symmetry, for the flows (3.9a,b), the force $\mathbf{F}$ on the sphere is normal to the surface, whilst the moment of force $\mathbf{G}$ is zero, and for the flows (3.10a,b) the force $\mathbf{F}$ is parallel to the surface (in the 2-direction) whilst the torque $\mathbf{G}$ is also parallel to the surface (in the 3- direction). Thus for the 'symmetric' flows (3.9a,b)

$$\mathbf{F} = (F_1, 0, 0) \qquad \mathbf{G} = \mathbf{0} \tag{3.11}$$

and for the 'antisymmetric' flows (3.10a,b)

$$\mathbf{F} = (0, F_2, 0) \qquad \mathbf{G} = (0, 0, G_3). \tag{3.12}$$

The values of the undisturbed velocity $\mathbf{U} = (U_1, U_2, 0)$ and pressure P in Cartesian coordinates (obtained using (3.1) and (3.6)) are listed in Table 1 for the flows (3.9 a,b) [up to $N = 6$] and in Table 2 for the flows (3. 10a,b) [up to $N = 5$].

3.2. Solid Surface

For a solid surface at $r_1 = 0$, the boundary conditions (2.11), when expressed in terms of the streamfunction ψ, reduce to

$$\psi = 0 \qquad \text{on both } \theta = -\frac{\pi}{2} \text{ and } \theta = +\frac{\pi}{2}, \tag{3.13}$$

$$\frac{\partial\psi}{\partial\theta} = 0 \qquad \text{on both } \theta = -\frac{\pi}{2} \text{ and } \theta = +\frac{\pi}{2}. \tag{3.14}$$

Table 3. Values of U_1, U_2, and P/μ in Cartesian coordinates for the 'symmetric' flows (3.15a, b) bounded by a free surface at $r_1 = 0$.

ψ	U_1	U_2	P/μ
$\rho^3(\sin 3\theta + \sin\theta)$	$4r_1^2$	$-8r_1r_2$	$8r_1$
$\rho^4(2\sin 4\theta + 4\sin 2\theta)$	$16r_1^3$	$-48r_1^2r_2$	$48(r_1^2 - r_2^2)$
$\rho^5(\sin 5\theta + \sin 3\theta)$	$8r_1^2(r_1^2 - 3r_2^2)$	$16r_1r_2(-2r_1^2 + r_2^2)$	$16r_1(r_1^2 - 3r_2^2)$
$\rho^6(4\sin 6\theta + 6\sin 4\theta)$	$48r_1^3(r_1^2 - 5r_2^2)$	$240r_1^2r_2(-r_1^2 + r_2^2)$	$120(r_1^4 - 6r_1^2r_2^2 + r_2^4)$

Substituting the value of ψ given by (3.5) in these boundary conditions and solving, we again obtain independent solutions for which ψ is an odd function of θ giving the 'symmetric' flows

$$\psi = \rho^N(\sin N\theta + \sin(N-2)\theta) \qquad \text{where } N = 3, 5, 7\ldots \tag{3.15a}$$

$$\psi = \rho^N((N-2)\sin N\theta + N\sin(N-2)\theta) \qquad \text{where } N = 4, 6, 8\ldots \tag{3.15b}$$

and independent solutions for which ψ is an even function of θ, giving the 'antisymmetric' flows

$$\psi = \rho^N(\cos N\theta + \cos(N-2)\theta) \qquad \text{where } N = 2, 4, 6\ldots \tag{3.16a}$$

$$\psi = \rho^N(N-2)\cos N\theta + N\cos(N-2)\theta) \qquad \text{where } N = 3, 5, 7\ldots. \tag{3.16b}$$

When the solid sphere is placed into each of these flows, it is again observed that, by symmetry, the force $\mathbf{F}$ and moment of force $\mathbf{G}$ on the sphere are of the forms

$$\mathbf{F} = (F_1, 0, 0) \qquad \mathbf{G} = \mathbf{0} \tag{3.17}$$

for the 'symmetric' flows (3.15a,b), and of the form

$$\mathbf{F} = (0, F_2, 0), \quad \mathbf{G} = (0, 0, G_3) \tag{3.18}$$

for the 'antisymmetric' flows (3.16a,b).

The values of the undisturbed velocity $\mathbf{U} = (U_1, U_2, 0)$ and pressure P for each of the flows (3.15a,b) [up to $N = 6$] are listed in Table 3 and for each of the flows (3.16 a,b) [up to $N = 5$] are listed in Table 4.

4. Forces acting on sphere placed in the flow (U, P)

Into each of the undisturbed flows given by (3.9a,b) and (3.10a,b), bounded by an undeformable free surface at $r_1 = 0$, we assume we place at rest a solid sphere of radius a with centre C at position $(h, 0, 0)$.The force $\mathbf{F}$ and the moment of force $\mathbf{G}$ (about C) acting on the sphere are then calculated using the results (2.25) and (2.35) with n replaced by $N - 1$. In order to perform this calculation we use values of ${}^F u_{Tik}$ and ${}^F u_{Rik}$ which are already known (see, for example, Brenner [6]) from the solution of the creeping flow equations (2.13)–(2.15) [or (2.27)–(2.29)] with the boundary conditions (2.16)–(2.18) [or (2.30)–(2.32)] obtained by

Table 4. Values of U_1, U_2, and P/μ in Cartesian coordinates for the 'antisymmetric' flows (3.16a, b) bounded by a free surface at $r_1 = 0$.

ψ	U_1	U_2	P/μ
$\rho^2(\cos 2\theta + 1)$	0	$-4r_1$	0
$\rho^3(\cos 3\theta + 3\cos\theta)$	0	$-12r_1^2$	$-24r_2$
$\rho^4(\cos 4\theta + \cos 2\theta)$	$-12r_1^2 r_2$	$4r_1(-2r_1^2 + 3r_2^2)$	$-24r_1r_2$
$\rho^5(3\cos 5\theta + 5\cos 3\theta)$	$-80r_1^3 r_2$	$40r_1^2(-r_1^2 + 3r_2^2)$	$80r_2(-3r_1^2 + r_2^2)$

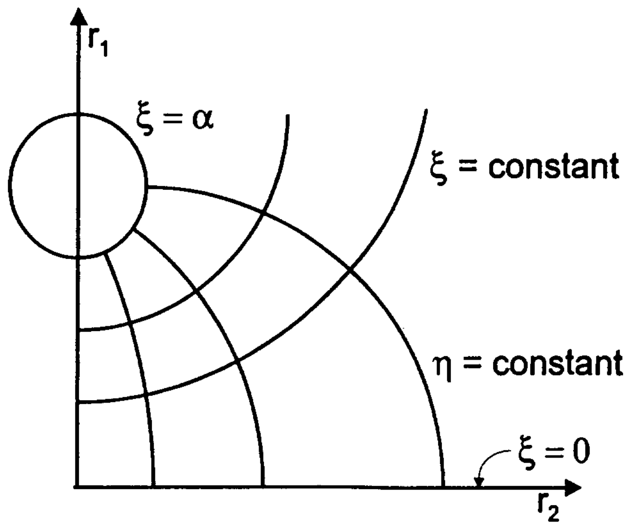


Fig. 4. The spherical bipolar coordinates (ξ, η, ϕ) are obtained by rotating the planar bipolar coordinates (ξ, η) shown about the r_1-axis.

the boundary conditions (2.16)–(2.18) [or (2.30)–(2.32)] obtained by using spherical bipolar coordinates (ξ, η, ϕ) defined in terms of the Cartesian coordinates (r_1, r_2, r_3) by

$$r_1 = \frac{c\sinh\xi}{\cosh\xi - \cos\eta}, \qquad \sqrt{r_2^2 + r_2^2} = \frac{c\sin\eta}{\cosh\xi - \cos\eta}, \qquad \frac{r_3}{r_2} = \tan\phi, \tag{4.1a,b,c}$$

so that the surfaces of the sphere and of the planar boundary $r_1 = 0$ are respectively $\xi = \alpha$ and $\xi = 0$ [see Fig. 4]. The values of the constants c and α expressed in terms of a and h are then

$$c = a\sinh\alpha = \sqrt{h^2 - a^2}, \tag{4.2a}$$

$$\alpha = \cosh^{-1}\left(\frac{h}{a}\right) = \log\left(\frac{h}{a} + \sqrt{\left(\frac{h}{a}\right)^2 - 1}\right). \tag{4.2b}$$

By calculating the asymptotic expansions of ${}^F u_{Tik}$ and ${}^F u_{Rik}$ for $r \to \infty$ (i.e. for $\xi \to 0, \eta \to 0$), the values of ${}_{(-N)}{}^F u_{Tik}$ and ${}_{(-N)}{}^F u_{Rik}$ may be obtained. From these the corresponding stress fields ${}_{(-N)}{}^F \sigma_{Tijk}$ and ${}_{(-N)}{}^F \sigma_{Rijk}$ are determined. These are then substituted respectively into (2.25) and (2.35) and the integrals evaluated algebraically to obtain the force $\mathbf{F} = (F_1, 0, 0)$ [with $\mathbf{G} = \mathbf{0}$] for the 'symmetric' flows (3.9a,b) and the force $\mathbf{F} = (0, F_2, 0)$ and moment of force $\mathbf{G} = (0, 0, G_3)$ for the 'antisymmetric' flows (3.10a,b).

This calculation, although straightforward, is long and tedious even for undisturbed flows with streamfunction ψ for which N is small. In fact, to do the calculation by hand for larger values of N is completely out of the question. The calculation was therefore performed on a computer using the symbolic manipulation language MACSYMA for all the flows (3.9) and (3.10) for which $N \leq 10$.

In a similar manner the force $\mathbf{F}$ and moment of force $\mathbf{G}$ (about C) on a sphere placed at rest in each of the flows (3.15a,b) and (3.16a,b) bounded by a solid wall at $r_1 = 0$ may be calculated using (2.39) and (2.40) in which ${}_{(-n)}{}^{S}u_{Tik}$, ${}_{(-n)}{}^{S}u_{Rik}$, ${}_{(-n)}{}^{S}\sigma_{Tijk}$ and ${}_{(-n)}{}^{S}\sigma_{Rijk}$ are obtained from the known values [4,6,8,18] of ${}^{S}u_{Tik}$ and ${}^{S}u_{Rik}$ given in terms of the spherical bipolar coordinates (ξ, η, ϕ). This calculation was again performed for all flows (3.15) and (3.16) with $N \leq 10$ using MACSYMA.

In the following sub-sections the results of these calculations are given. The force $\mathbf{F}$ and moment of force $\mathbf{G}$ acting on the sphere in the 'symmetric' flow (3.9a,b) and 'antisymmetric' flow (3.10a,b) for an undeformable free surface at $r_1 = 0$ are given in sub-sections 4.1 and 4.2 respectively whilst the equivalent results for the 'symmetric' flow (3.15a,b) and 'antisymmetric' flow (3.16a,b) bounded by a solid wall at $r_1 = 0$ are given in sub-sections 4.3 and 4.4.

4.1. 'SYMMETRIC' FLOW BOUNDED BY A FREE SURFACE

We consider undisturbed 'symmetric' flows (3.9a,b) bounded by an undeformable free surface at $r_1 = 0$ with a general (dimensional) strength S so that ψ is now

$$\psi = S\rho^N \sin N\theta \qquad (N = 2, 4, 6 \ldots) \tag{4.3a}$$

or

$$\psi = S\rho^N \sin(N-2)\theta \qquad (N = 4, 6, 8 \ldots). \tag{4.3b}$$

If $\mathbf{F} = (F_1, 0, 0)$ is the dimensional force on the sphere placed at rest in such flows, we define the dimensionless force $\mathbf{F}^* = (F_1^*, 0, 0)$ as

$$\mathbf{F}^* = \frac{\mathbf{F}}{6\pi\mu a^N S}. \tag{4.4}$$

Then by making use of the value of ${}^{F}u_{Ti1}$ given by Brenner [6], the calculation described above using MACSYMA gives the value of F_1^* for both of the flow types (4.3a) and (4.3b) as

$$F_1^* = +\frac{\sqrt{2}}{6} \sinh^{N-2}\alpha \sum_{s=1}^{\infty} \left\{ \left(s - \frac{1}{2}\right) ({}_bK_s) b_s + \left(s + \frac{3}{2}\right) ({}_dK_s) d_s \right\}, \tag{4.5}$$

where α is given by (4.2b), and b_s and d_s are

$$b_s = -\frac{s(s+1)\sinh^2\alpha}{\sqrt{2}(2s-1)} \left[\frac{2(1 + e^{-(2s+1)\alpha}) + (2s+1)(e^{2\alpha} - 1)}{2\sinh(2s+1)\alpha - (2s+1)\sinh 2\alpha} \right], \tag{4.6a}$$

Table 5. Values of $L, a_0, a_1 \ldots$ in (4.7) for ${}_bK_s$ and ${}_dK_s$ (see 4.5)) for the flows (4.3a) with $N \leq 10$. In the second column b and d indicate whether the values are for ${}_bK_s$ or ${}_dK_s$.

N	${}_bK_s$ or ${}_dK_s$	L	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
2	b	8	−1								
	d	8	−1								
4	b	$\frac{16}{15}$	−9	+1	−7						
	d	$\frac{16}{15}$	−17	−15	−7						
6	b	$\frac{8}{105}$	−135	+26	−188	+4	−22				
	d	$\frac{8}{105}$	−375	−502	−332	−92	−22				
8	b	$\frac{32}{945}$	−315	+81	−601	+23	−129	+1	−5		
	d	$\frac{32}{945}$	−1155	−1903	−1529	−649	−209	−31	−5		
10	b	$\frac{8}{31185}$	−42525	+13212	−100152	+5252	−29942	+428	−2168	+8	−38
	d	$\frac{8}{31185}$	−193725	−364548	−333592	−175068	−67542	−15732	−3288	−312	−38

$$d_s = +\frac{s(s+1)\sinh^2\alpha}{\sqrt{2}(2s+3)}\left[\frac{2(1+e^{-(2s+1)\alpha})+(2s+1)(1-e^{-2\alpha})}{2\sinh(2s+1)\alpha-(2s+1)\sinh 2\alpha}\right], \tag{4.6b}$$

which are (apart from a factor of a^2) quantities appearing in the value of ${}^{F}u_{Ti1}$ given by Brenner [6]. The quantities ${}_bK_s$ and ${}_dK_s$ appearing in (4.5) are polynomials of degree ($N-2$) in s and are of the form

$$\begin{matrix} {}_bK_s \\ {}_dK_s \end{matrix} = L\left(a_0 + a_1 s + \ldots + a_{N-2}s^{N-2}\right) \tag{4.7}$$

where L is a rational number and $a_0, a_1 \ldots a_{N-2}$ are integers whose values depend on N and on whether the flow is of the type given by (4.3a) or (4.3b). These values of L, a_0, $a_1 \ldots$ for both ${}_bK_s$ and ${}_dK_s$ for $N \leq 10$ are listed in Table 5 for flows given by (4.3a) and in Table 6 for flows given by (4.3b).

Thus for the flows (4.3a,b) with $N \leq 10$ one may, to any desired accuracy, calculate numerically the dimensionless force F_1^* on the sphere as a function of h/a using (4.5). This has been done, and in order to illustrate the results so obtained it is convenient to define a new dimensionless force $\overline{F}_1^*$ and a dimensionless gap distance h^* ($0 \leq h^* < \infty$) between sphere and plane $r_1 = 0$ as

$$\overline{F}_1^* = \frac{F_1}{6\pi\mu a U_{C1}}, \tag{4.8}$$

$$h^* = \frac{h-a}{\alpha}, \tag{4.9}$$

with U_{Ci} = being the ith component of the undisturbed velocity U_C evaluated at the sphere centre C. With these definitions we have $\overline{F}_1^* \to 1$ as $h^* \to \infty$ since the effect of the boundary at $r_1 = 0$ would then become negligible. It may be readily shown that

$$\overline{F}_1^* = \frac{F_1^*}{N(1+h^*)^{N-1}} = \frac{F_1^*}{N\cosh^{N-1}\alpha} \tag{4.10a}$$

Table 6. Values of $L, a_0, a_1 \dots$ in (4.7) for ${}_bK_s$ and ${}_dK_s$ (see 4.5)) for the flows (4.3b) with $N \leq 10$. In the second column b and d indicate whether the values are for ${}_bK_s$ or ${}_dK_s$.

N	${}_bK_s$ or ${}_dK_s$	L	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
4	b	$\frac{8}{15}$	+3	−17	−1						
	d	$\frac{8}{15}$	+19	+15	−1						
6	b	$\frac{16}{315}$	+45	−284	+2	−76	−2				
	d	$\frac{16}{315}$	+405	+508	+218	+68	−2				
8	b	$\frac{8}{945}$	+315	−2151	+115	−1037	−9	−67	−1		
	d	$\frac{8}{945}$	+3675	+5785	+3827	+1651	+311	+61	−1		
10	b	$\frac{32}{155925}$	+14175	−102942	+9162	−68242	+352	−8158	−62	−208	−2
	d	$\frac{32}{155925}$	+203175	+369258	+300962	+157158	+47352	+12042	+1338	+192	−2

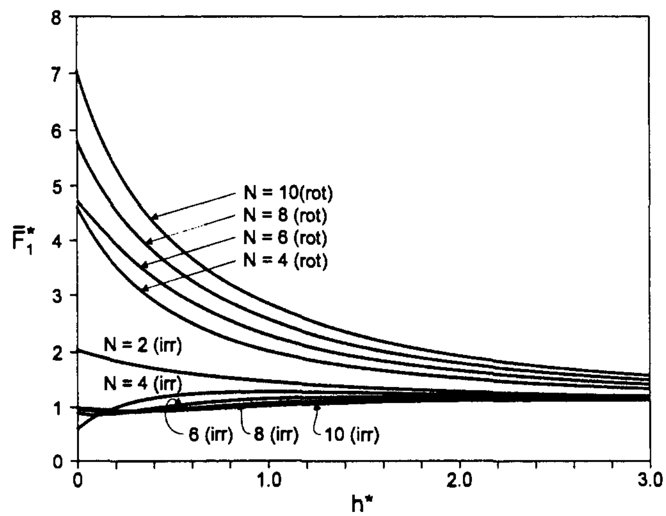


Fig. 5. Force $\overline{F}_1^*$ as a function of h^* for 'symmetric' flows bounded by a free surface at $r_1 = 0$. The values of N are indicated with(*irr*) and (*rot*) referring to whether the flow is irrotational (given by(4.3a)) or rotational (given by (4.3b)).

for the flows (4.3a), and

$$\overline{F}_1^* = \frac{F_1^*}{(N-2)(1+h^*)^{N-1}} = \frac{F_1^*}{(N-2)\cosh^{N-1}\alpha} \tag{4.10b}$$

for the flows (4.3b), so that $\overline{F}_1^*$ known once F_1^* has been calculated.

The results for the dimensionless force $\overline{F}_1^*$ acting on the sphere as a function of h^* have been plotted in Fig. 5 for all the flows (4.3a) and (4.3b) with $N \leq 10$.

Finally, it should be noted from Brenner [6] that for the sphere translating (but not rotating) with velocity $\hat{U}$ in the 1-direction in a quiescent fluid bounded by the free surface, the dimensionless force $\overline{F}_1^*$ on the sphere in the 1-direction (made dimensionless by $6\pi\mu a\hat{U}$ so that $\overline{F}_1^* \to -1$ as $h^* \to \infty$) is

$$\overline{F}_1^* = \frac{\sqrt{2}}{3\sinh\alpha}\sum_{s=1}^{\infty}(b_s + d_s). \tag{4.11}$$

4.2. 'ANTISYMMETRIC' FLOW BOUNDED BY A FREE SURFACE

Consider the undisturbed 'antisymmetric' flows (3.10 a,b) bounded by an undeformable free surface at $r_1 = 0$ with a general strength S so that ψ is now

$$\psi = S\rho^N \cos N\theta \qquad (N = 3, 5, 7\ldots) \tag{4.12a}$$

and

$$\psi = S\rho^N \cos(N-2)\theta \qquad (N = 3, 5, 7\ldots). \tag{4.12b}$$

For a sphere placed at rest in such flows (with centre on the r_1-axis) the dimensional force **F** = $(0, F_2, 0)$ and moment of force **G** = $(0, 0, G_3)$ about the sphere centre C may be expressed in terms of a dimensionless force **F** = $(0, F_2^*, 0)$ and moment of force $\mathbf{G}^*$ = $(0, 0, G_3^*)$ by

$$\mathbf{F}^* = \frac{\mathbf{F}}{6\pi\mu a^N S}, \qquad \mathbf{G}^* = \frac{\mathbf{G}}{8\pi\mu a^{N+1} S}. \tag{4.13}$$

The calculated flows obtained by Goldman *et al.* [10] for a pair of equal sized spheres translating or rotating in a symmetric manner in a quiescent fluid may be used for the present problem to obtain ${}^F u_{Ti2}$ and ${}^F u_{Ri3}$ for a single sphere translating or rotating in a fluid bounded by an undeformable free surface, since on the plane of symmetry for the two-sphere problem the free surface boundary condition is automatically satisfied. From these we calculate, using MACSYMA, in the manner described above, the values of F_2^* and G_3^* for both of the flow types (4.12a) and (4.12b) as

$$\begin{aligned} F_2^* &= +\frac{\sqrt{2}}{6}\sinh^N\alpha\sum_{s=0}^{\infty}\{s({}_aK_S)^T a_s + s(s+1)({}_bK_s)^T b_s \\ &\qquad + ({}_dK_s)^T d_s + s(s-1)({}_fK_s)^T f_s\}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} G_3^* &= +\frac{\sqrt{2}}{8}\sinh^{N+1}\alpha\sum_{s=0}^{\infty}\{s({}_aK_s)^R a_s + s(s+1)({}_bK_s)^R b_s + ({}_dK_s)^R d_s \\ &\qquad + s(s-1)({}_fK_s)^R f_s\}, \end{aligned} \tag{4.15}$$

where α is again given by (4.2b). In (4.14) the values of ${}^T a_s$, ${}^T b_s$, ${}^T d_s$, and ${}^T f_s$ are quantities appearing in the value of ${}^F u_{Ti2}$ determined by Goldman *et al.* [10] with ${}^T a_s$ as being determine by the infinite set of equations

$$\left[(s-1)(\gamma_{s-1}) - \frac{(s-1)(2s-3)}{(2s-1)}(\gamma_s - 1)\right]{}^T a_{s-1}$$

$$+\left[(2s+1)-5\gamma_s-\frac{s(2s-1)}{(2s+1)}(\gamma_{-1}+1)+\frac{(s+1)(2s+3)}{(2s+1)}(\gamma_{s-1}-1)\right]{}^{T}a_s$$

$$+\left[\frac{(s+2)(2s+5)}{(2s+3)}(\gamma_s+1)-(s+2)(\gamma_{s+1}+1)\right]{}^{T}a_{s+1}$$

$$=\sqrt{2}e^{-(s+\frac{1}{2})\alpha}\left[\frac{e^{\alpha}}{\cosh(s-\frac{1}{2})\alpha}-\frac{2}{\cosh(s+\frac{1}{2})\alpha}+\frac{e^{-\alpha}}{\cosh(s+\frac{3}{2})\alpha}\right]\quad (s\leqslant 1)\ (4.16)$$

for ${}^{T}a_1, {}^{T}a_2$ ... in which γ_s is

$$\gamma_s=\coth\alpha\tanh(s+\frac{1}{2})\alpha. \tag{4.17}$$

${}^{T}b_s$, ${}^{T}d_s$ and ${}^{T}f_s$ are then given in terms of ${}^{T}a_s$ by

$$\begin{aligned}{}^{T}b_s = &\left[2\left(\frac{s-1}{2s-1}\right)(\gamma_s-1)\right]{}^{T}a_{s-1}\\ &-[2\gamma_s]^{T}a_s+\left[2\left(\frac{s+2}{2s+3}\right)_{(\gamma_s+1)}\right]{}^{T}a_{s+1},\quad (s\geq 1),\end{aligned} \tag{4.18}$$

$$\begin{aligned}{}^{T}d_s=2\sqrt{2}e^{-(s+\frac{1}{2})\alpha}\operatorname{sech}(s+\frac{1}{2})\alpha\quad &-\left[\frac{s(s-1)}{(2s-1)}(\gamma_s-1)\right]{}^{T}a_{s-1}\\ &+\left[\frac{(s+1)(s+2)}{(2s+3)}\gamma_s+1\right]{}^{T}a_{s+1},\quad (s\geq 0),\end{aligned} \tag{4.19}$$

$${}^{T}f_s=\left[\frac{(\gamma_s-1)}{(2s-1)}\right]^{T}a_{s-1}-\left[\frac{(\gamma_s-1)}{(2s-1)}\right]^{T}a_{s-1}-\left[\frac{(\gamma_s+1)}{(2s+3)}\right]^{T}a_{s+1}\quad (s\geq 2). \tag{4.20}$$

The values of ${}^{T}a_0$, ${}^{T}b_0$, ${}^{T}f_0$ and ${}^{T}f_1$ are not defined by the above Eqs. (4.16)–(4.20) but they are not needed anyway since they do not affect the value of F_2^* given by (4.14).

Likewise, in (4.15), ${}^{R}a_s$, ${}^{R}b_s$, ${}^{R}d_s$ and ${}^{R}f_s$ are quantities appearing in the value of ${}^{F}u_{Ri3}$ determined by Goldman *et al.* [10] with ${}^{R}a_s$ being determined by the infinite set of equations

$$\left[(s-1)(\gamma_s-1)-\frac{(s-1)(2s-3)}{(2s-1)}(\gamma_s-1)\right]{}^{R}a_{s-1}$$

$$+\left[(2s+1)-5\gamma_s-\frac{s(2s-1)}{(2s+1)}(\gamma_{s-1}+1)+\frac{(s+1)(2s+3)}{(2s+1)}(\gamma_{s+1}-1)\right]{}^{R}a_s$$

$$+\left[\frac{(s+2)(2s+5)}{(2s+3)}(\gamma_s+1)-(s+2)(\gamma_{s+1}+1)\right]{}^{R}a_{s+1}$$

$$\begin{aligned}&=\frac{\sqrt{2}e^{-(s+\frac{1}{2})\alpha}}{\sinh\alpha}\left[\frac{(2s+1)}{\cosh(s+\frac{1}{2})\alpha}\left(\frac{e^{\alpha}}{2s-1}+\frac{e^{-\alpha}}{2s+3}\right)\right.\\ &\left.-\frac{(2s-1)}{(2s+1)\cosh(s-\frac{1}{2})\alpha}-\frac{(2s+3)}{(2s+1)\cosh(s+\frac{3}{2})\alpha}\right],\quad (s\geq 1),\end{aligned} \tag{4.21}$$

for ${}^{R}a_1, {}^{R}a_2\ldots$; ${}^{R}b_s$, ${}^{R}d_s$ and ${}^{R}f_s$ are then given in terms of ${}^{R}a_s$ by

$${}^{R}b_s=\frac{4\tau_s}{\sinh\alpha\cosh(s+\frac{1}{2})\alpha}$$

$$+\left[2\frac{(s-1)}{(2s-1)}(\gamma_s-1)\right]{}^{R}a_{s-1}-[2\gamma_s]\,{}^{R}a_s+\left[2\frac{(s+2)}{(2s+3)}(\gamma_s+1)\right]{}^{R}a_{s+1},\quad (s\geq 1),\tag{4.22}$$

$$\begin{aligned}{}^{R}d_s &= \frac{2\sqrt{2}}{\sinh\alpha\cosh(s+\frac{1}{2})\alpha}\left[\frac{s^2}{(2s-1)}e^{-(s-\frac{1}{2})\alpha}-\frac{(s+1)^2}{(2s+3)}e^{-(s+\frac{3}{2})\alpha}\right]\\ &\quad -\left[\frac{s(s-1)}{(2s-1)}(\gamma_s-1)\right]{}^{R}a_{s-1}+\left[\frac{(s+1)(s+2)}{(2s+3)}(\gamma_s+1)\right]{}^{R}a_{s+1},\quad (s\geq 0),\end{aligned}\tag{4.23}$$

and

$${}^{R}f_s=-\frac{4\tau_s}{\sinh\alpha\cosh(s+\frac{1}{2})\alpha}+\left[\frac{(\gamma_s-1)}{(2s-1)}\right]{}^{R}a_{s-1}-\left[\frac{(\gamma_s+1)}{(2s+3)}\right]{}^{R}a_{s+1},\quad (s\geq 2),\tag{4.24}$$

where τ_s is defined as

$$\tau_s=-\frac{1}{\sqrt{2}}\left[\frac{e^{-(s-\frac{1}{2})\alpha}}{2s-1}-\frac{e^{-(s+\frac{3}{2})\alpha}}{2s+3}\right].\tag{4.25}$$

Again ${}^{R}a_0$, ${}^{R}b_0$, ${}^{R}f_0$, and ${}^{R}f_1$ are not defined and are not needed to determine G_3^* from (4.15).

The quantities ${}_aK_s$, ${}_bK_s$, ${}_dK_s$, and ${}_fK_s$ appearing in the results (4.14) [for the dimensionless force F_2^*] and (4.15) [for the dimensionless moment of force G_3^*] are all polynomials of degree $(N-1)$ in s and are of the form

$$-aK_s=L(a_0+a_1s+\ldots+a_{N-1}s^{N-1}),\tag{4.26}$$

where the value of L depends on the value of N and on whether the undisturbed flow is of type (4.12a) or of type (4.12b). The values of $a_0, a_1, \ldots$ are however found to depend only on N [being the same for flow (4.12b) as for flow (4.12a)]. These values, for $N\leq 10$, have been listed in Table 7. Using these values, the dimensionless force F_2^* and moment of force G_3^* on the sphere were calculated numerically as a function of h/a using (4.14) and (4.15). To illustrate the results so obtained, we define a new dimensionless force $\overline{F}_2^*$ in a manner similar to (4.8) as

$$\overline{F}_2^*=\frac{F_2}{6\pi\mu a U_{C2}},\tag{4.27}$$

so that $\overline{F}_2^*\to 1$ as $h^*\to\infty$. $\overline{F}_2^*$ is then related to F_2^* for both flow types (4.12a) and (4.12b) by

$$\overline{F}_2^*=\frac{F_2^*}{N\cosh^{N-1}\alpha}.\tag{4.28}$$

Also a new dimensionless moment of force $\overline{G}_3^*$ (about C) is defined as

$$\overline{G}_3^*=\frac{G_3}{4\pi\mu a^3\omega_{C3}}\tag{4.29}$$

for flow type (4.12b) where ω_{Ci} is the ith component of ω_C, the vorticity $\nabla\times\mathbf{U}$ of the undisturbed flow evaluated at the sphere centre C. However, (4.29) cannot be used to define

Table 7. Values of $L, a_0, a_1 \ldots$ in (4.26) for ${}_aK_s, {}_bK_s, {}_dK_s$ and ${}_fK_s$ ((see 4.14) and (4.15) for the flows (4.12a,b)) with $N \leq 10$. In the second column *a, b, d,* and *f* indicate whether the values for ${}_aK_s, {}_bK_s, {}_dK_s$ or ${}_fK_s$. L_{irr} and L_{rot} are respectively the values of L for the irrotational flow (4.12a) and the rotational flow (4.12b)

N	a, b d, f	L_{irr}	L_{rot}	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
3	a	$\frac{40}{7}$	$-\frac{248}{105}$	-1	-3	-2						
	b	$\frac{24}{7}$	$+\frac{8}{105}$	-1	-1	-1						
	d	$\frac{30}{7}$	$+\frac{78}{35}$	-1	-2	-2						
	f	$\frac{10}{7}$	$+\frac{26}{35}$	2	3	1						
5	a	$\frac{128}{33}$	$-\frac{416}{495}$	-3	-10	-10	-5	-2				
	b	$\frac{70}{99}$	$+\frac{14}{495}$	-9	-14	-16	-4	-2				
	d	$\frac{80}{33}$	$+\frac{80}{99}$	-3	-8	-10	-4	-2				
	f	$\frac{32}{33}$	$+\frac{32}{99}$	6	13	13	8	2				
7	a	$\frac{88}{225}$	$-\frac{232}{4095}$	-45	-161	-196	-140	-70	-14	-4		
	b	$\frac{28}{135}$	$+\frac{20}{2457}$	-45	-87	-109	-45	-25	-3	-1		
	d	$\frac{154}{675}$	$+\frac{686}{12285}$	-45	-138	-196	-120	-70	-12	-4		
	f	$\frac{22}{45}$	$+\frac{98}{819}$	18	47	61	47	19	5	1		
9	a	$\frac{448}{5985}$	$-\frac{7424}{915705}$	-315	-1188	-1636	-1386	-798	-252	-84	-9	-2
	b	$\frac{234}{29925}$	$+\frac{1274}{4578525}$	-1575	-3492	-4688	-2492	-1498	-308	-112	-8	-2
	d	$\frac{252}{5985}$	$+\frac{7452}{915705}$	-315	-1056	-1636	-1232	-798	-224	-84	-8	-2
	f	$\frac{784}{29925}$	$+\frac{23184}{4578525}$	450	1323	1963	1713	873	327	87	12	2

$\overline{G}_3^*$ for flow type (4.12a) which is an irrotational flow giving $|\omega_C| = 0$. Instead for flow type (4.12a), we define $\overline{G}_3^*$ as

$$\overline{G}_3^* = \frac{G_3}{4\pi\mu a^3 (U_{C2}/h)}. \tag{4.30}$$

Thus $\overline{G}_3^* \rightarrow 0$ for flow type (4.12a) and $\overline{G}_3^* \rightarrow 1$ for flow type (4.12b) as $h^* \rightarrow \infty$. $\overline{G}_3^*$ is related to G_3^* by

$$\overline{G}_3^* = -\frac{2}{N \cosh^{N-2} \alpha} G_3^* \tag{4.31}$$

for flow type (4.12a), and by

$$\overline{G}_3^* = -\frac{1}{2(N-1) \cosh^{N-2} \alpha} G_3^* \tag{4.32}$$

for flow type (4.12b).

The results for the dimensionless force $\overline{F}_2^*$ and moment of force $\overline{G}_3^*$ acting on the sphere as a function of h^* have been plotted respectively in Figs. 6a and 6b for all flows (4.12a,b) with $N \leq 10$.

Finally, it is seen directly from Goldman *et al.* [10] that for the sphere translating (but not rotating) with velocity $\hat{U}$ in the 2-direction in a quiescent fluid bounded by a free surface, the

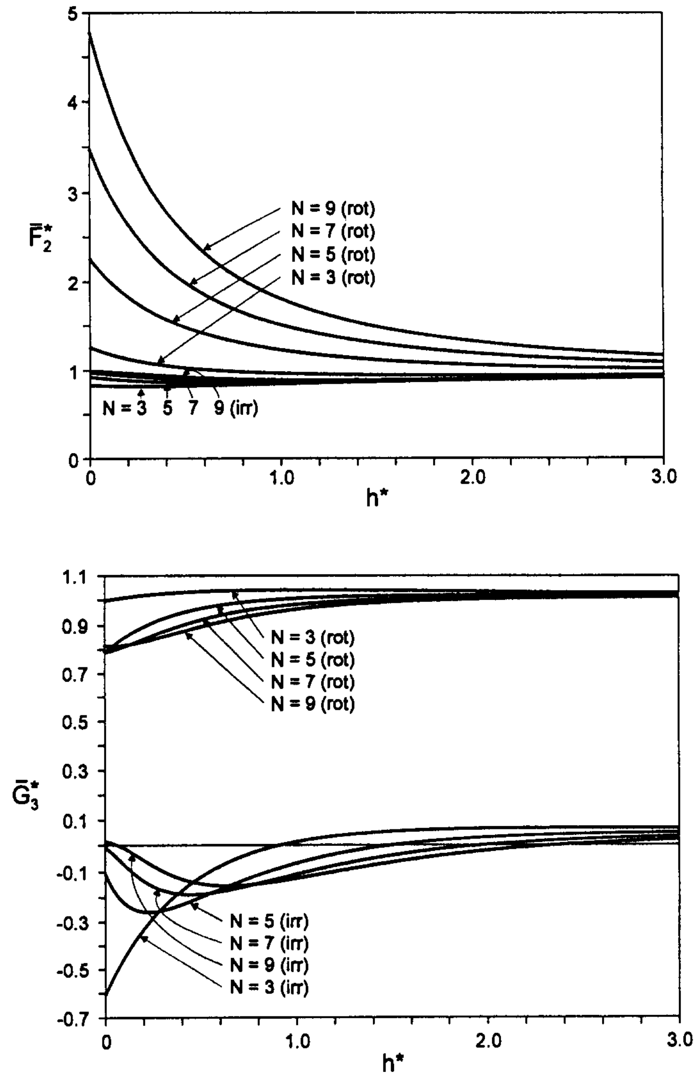


Fig. 6. Force $\overline{F_2^*}$ shown in (a) and moment of force $\overline{G_3^*}$ shown in (b) as a function of h^* for 'antisymmetric' flows bounded by a free surface at $r_1 = 0$. The values of N are indicated with (*irr*) and (*rot*) referring to whether the flow is irrotational (given by (4.12a)) or rotational (given by (4.12b)). Note that for the flows (4.12a), $\overline{G_3^*}$ is non-dimensionalised as in (4.30) whereas for flows (4.12b), it is non-dimensionalised as in (4.29).

dimensionless force $\overline{F_2^*}$ on the sphere in the 2-direction (made dimensionless by $6\pi\mu a^2\hat{U}$ so that $\overline{F_2^*} \to -1$ as $R^* \to \infty$) is

$$\overline{F_2^*} = -\frac{\sqrt{2}}{6}\sinh\alpha\sum_{s=0}^{\infty}\left\{s(s+1)^T b_s + ^T d_s\right\}, \tag{4.33}$$

whilst the dimensionless moment of force $\overline{G_3^*}$ on the sphere about its center in the 3-direction (made dimensionless by $8\pi\mu a^2\hat{U}$ so that $\overline{G_3^*} \to 0$ as $h^* \to \infty$) is

$$\overline{G_3^*} = \frac{\sinh^2\alpha}{12\sqrt{2}}\sum_{s=0}^{\infty}\left\{2s(s+1)\left[2 + e^{-(2s+1)\alpha}\right] {}^T a_s\right.$$

$$+[2 - e^{-(2s+1)\alpha}][s(s+1)(\coth\alpha)^T b_s - (2s+1-\coth\alpha)^T d_s]\Big\}. \quad (4.34)$$

Likewise from Goldman *et al.* [10], for the sphere rotating about its centre (but not translating) with angular velocity $\hat{\Omega}$ in the 3-direction in a quiescent fiuid bounded by a free surface, the dimensionless force $\overline{F}_2^*$ on the sphere in the 2-direction (made dimensionless by $6\pi\mu a^2\hat{\Omega}$ so that $\overline{F}_2^* \to 0$ as $h^* \to \infty$) is

$$\overline{F}_2^* = -\frac{\sqrt{2}}{6}\sinh^2\alpha \sum_{s=0}^{\infty}\left\{s(s+1)^R b_s + {}^R d_s\right\}, \quad (4.35)$$

whilst the dimensionless moment of force $\overline{G}_3^*$ on the sphere about its centre in the 3-direction (made dimensionless by $8\pi\mu a^3\hat{\Omega}$ so that $\overline{G}_3^* \to -1$ as $h^* \to \infty$) is

$$\begin{aligned}\overline{G}_3^* = & -\frac{1}{3} + \frac{\sinh^3\alpha}{12\sqrt{2}}\sum_{s=0}^{\infty}\Big\{2s(s+1)\left[2 + e^{-(2s+1)\alpha)}\right]{}^R a_s \\ & + \left[2 - e^{(-2s+1)\alpha}\right]\left[s(s+1)(\coth\alpha)^R b_s - (2s+1-\coth\alpha)^R d_s\right]\Big\}.\end{aligned} \quad (4.36)$$

4.3. 'SYMMETRIC' FLOW BOUNDED BY A SOLID SURFACE

The undisturbed 'symmetric' flows (3.15 a,b) bounded by a solid surface at $r_1 = 0$ with a general (dimensional) strength S may be written as

$$\psi = S\rho^N(\sin N\theta + \sin(N-2)\theta) \text{ where } N = 3, 5, 7\ldots \quad (4.37a)$$

and

$$\psi = S\rho^N\left((N-2)\sin N\theta + N\sin(N-2)\theta\right) \text{ where } N = 4, 6, 8\ldots \quad (4.37b)$$

If $\mathbf{F}^* = (F_1^*, 0, 0)$ is the dimensional force on the sphere placed at rest (with centre at $(h, 0, 0)$), we define, as in §4.1, the dimensionless force $\mathbf{F}_1^* = (F_1^*, 0, 0)$ as

$$\mathbf{F}_1^* = \frac{\mathbf{F}}{6\pi\mu a^N S}. \quad (4.38)$$

Then by making use of the value of ${}^S u_{Ti1}$ given by Brenner [6], the value of F_1^* given by MACSYMA is obtained as

$$F_1^* = +\frac{\sqrt{2}}{6}\sinh^{N-2}\alpha\sum_{s=1}^{\infty}(s+\frac{1}{2})({}_cK_s)c_s \quad (4.39a)$$

for the flows (4.37a) for which N is odd, and as

$$F_1^* = +\frac{\sqrt{2}}{6}\sinh^{N-2}\alpha\sum_{s=1}^{\infty}(s^2-\frac{1}{4})({}_bK_s)b_s \quad (4.39b)$$

for the flows (4.37b) for which N is even. In (4.39a,b), c_s and b_s are (apart from a dimensional factor and sign) as given by Brenner [6], i.e.

$$c_s = -\frac{s(s+1)(2s+1)\sinh^4\alpha}{\sqrt{2}\left[4\sinh^2\left((s+\frac{1}{2})\alpha - (2s+1)^2\sinh^2\alpha\right]\right.}, \quad (4.40a)$$

Table 8. Values of $L, a_0, a_1 \ldots$ in (4.41) for ${}_cK_s$ and ${}_bK_s$ (see (4.39)) for the flows (4.37a, b) with $N \leq 10$. In the second column c and b indicate whether the values are for ${}_cK_s$ or ${}_bK_s$.

N	${}_cK_s$ or ${}_bK_s$	L	a_0	a_1	a_2	a_3	a_4	a_5	a_6
3	c	$-\frac{64}{3}$	1						
4	b	$-\frac{512}{15}$	1						
5	c	$-\frac{128}{15}$	3	1	1				
6	b	$-\frac{512}{35}$	5	1	1				
7	c	$-\frac{64}{105}$	45	26	28	4	2		
8	b	$-\frac{1024}{945}$	105	38	40	4	2		
9	c	$-\frac{256}{2835}$	315	243	277	69	37	3	1
10	b	$-\frac{1024}{6237}$	945	471	517	93	49	3	1

$$b_s = -\frac{s(s+1)\sinh^2\alpha}{\sqrt{2}(2s-1)}\left[\frac{2\sinh(2s+1)\alpha + (2s+1)\sinh 2\alpha}{4\sinh^2(s+\frac{1}{2})\alpha - (2s+1)^2\sinh^2\alpha} - 1\right]. \tag{4.40b}$$

The quantities ${}_cK_s$ and ${}_bK_s$ appearing in (4.39a,b) are polynomials in s of degree $(N-2)$ and $(N-3)$ respectively, and are of the forms

$$ {}_cK_s = L\left(a_0 + a_1 s + \ldots + a_{N-2}s^{N-2}\right),$$

$$ {}_bK_s = L\left(a_0 + a_1 s + \ldots + a_{N-3}s^{N-3}\right), \tag{4.41}$$

with the values of $L, a_0, a_l, \ldots$ for ${}_cK_s$ and for ${}_bK_s$ for all cases $N \leq 10$ being listed in Table 8.

Thus for the flows (4.37a,b) with $N \leq 10$, one may obtain the dimensionless force F_1^* on the sphere as a function of h/a by making use of the results (4.39), (4.40), (4.41) and Table 8. As in §4.1, it is more convenient to use a new dimensionless force $\overline{F}_1^*$ defined as

$$\overline{F}_1^* = \frac{F_1}{6\pi\mu a U_{C1}} \tag{4.42}$$

and gap distance h^* defined as in (4.9). The definition (4.42) for $\overline{F}_1^*$ with $\mathbf{U}_C$ being the undisturbed velocity evaluated at the sphere centre C means that $\overline{F}_1^* \to 1$ as $h^* \to \infty$. It may readily be shown that

$$\overline{F}_1^* = \frac{F_1^*}{2(N-1)(1+h^*)^{N-1}} = \frac{F_1^*}{2(N-1)\cosh^{N-1}\alpha}, \qquad (N = 3, 5, 7\ldots), \tag{4.43a}$$

for the flow (4.32a) and that

$$\overline{F}_1^* = \frac{F_1^*}{2N(N-2)(1+h^*)^{N-1}} = \frac{F_1^*}{2(N-2)\cosh^{N-1}\alpha}, \qquad (N = 4, 6, 8\ldots) \tag{4.43b}$$

for the flow (4.32b).

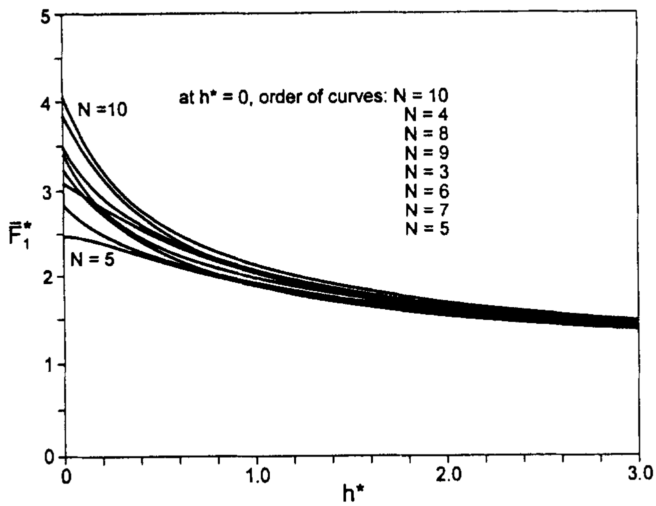


Fig. 7. $\overline{F}_1^*$ as a function of h^* for 'symmetric' flows bounded by a solid surface at $r_1 = 0$. The flows are given by (4.37a) when N is odd and by (4.37b) when N is even.

The results for the dimensionless force $\overline{F}_1^*$ acting on the sphere as a function of h^* have been plotted in Fig. 7 for all the flows (4.37a,b) with $N \leq 10$.

Finally, it should be noted from Brenner [6] that for the sphere translating (without rotation) with unit velocity $\hat{U}$ in the 1-direction in a quiescent fluid bounded by a solid surface, the dimensionless force $\overline{F}_1^*$ on the sphere in the 1- direction (made dimensionless by $6\pi\mu a\hat{U}$ so that $\overline{F}_1^* \to -1$ as $h^* \to \infty$) is

$$\overline{F}_1^* = \frac{\sqrt{2}}{3\sinh\alpha}\sum_{s=1}^{\infty}\frac{4b_s}{(2s+3)}. \tag{4.44}$$

4.4. 'ANTISYMMETRIC' FLOW BOUNDED BY A SOLID SURFACE

The undisturbed 'antisymmetric' flows (3.16a,b) bounded by a solid surface at $r_1 = 0$ with a general (dimensional) strength S may be written as

$$\psi = S\rho^N\left(\cos N\theta + \cos(N-2)\theta\right) \qquad \text{where } N = 2, 4, 6\ldots \tag{4.45a}$$

$$\psi = S\rho^N\left((N-2)\cos N\theta + N\cos(N-2)\theta\right) \qquad \text{where } N = 3, 5, 7\ldots \tag{4.45b}$$

The dimensional force $\mathbf{F} = (0, F_2, 0)$ and moment of force $\mathbf{G} = (0, 0, G_3)$ about C acting on a sphere (with centre on the r_1 - axis) may be expressed as in §4.2 in terms of a dimensionless force $\mathbf{F}^* = (0, F_2^*, 0)$ and moment of force $\mathbf{G}^* = (0, 0, G_3^*)$ by

$$\mathbf{F}^* = \frac{\mathbf{F}}{6\pi\mu a^N S}, \qquad \mathbf{G}^* = \frac{\mathbf{G}}{8\pi\mu a^{N+1} S}. \tag{4.46}$$

The values of ${}^{S}u_{Ti2}$ and ${}^{S}u_{Ri3}$ for a sphere translating or rotating in a quiescent fluid bounded by a solid wall and obtained by O'Neill [4,18] and by Dean & O'Neill [8] may, by using MACSYMA (and the results of §2), be used to obtain the values of F_2^* and G_3^* for flows of the type (4.45a) for N even as

$$F_2^* = +\frac{\sqrt{2}}{6}\sinh^N\alpha\sum_{s=0}^{\infty}(2s+1)\{s(s+1)({}_cK_s)^T c_s + ({}_eK_s)^T e_s$$
$$+s(s-1)({}_gK_s)^T g_s\}, \tag{4.47a}$$

$$G_3^* = +\frac{\sqrt{2}}{8}\sinh^{N+1}\alpha\sum_{s=0}^{\infty}(2s+1)\{s(s+1)({}_cK_s)^R c_s + ({}_eK_s)^R e_s + s(s-1)({}_gK_s)^R g_s\} \tag{4.48a}$$

and for flows of the type (4.45b) for N odd as

$$F_2^* = +\frac{\sqrt{2}}{6}\sinh^N\alpha\sum_{s=0}^{\infty}\{s(s+1)(2s+1)({}_aK_s)^T a_s + s(s+1)({}_bK_s)^T b_s + ({}_dK_s)^T d_s$$
$$+s(s-1)(s+1)(s+2)({}_fK_s)^T f_s\}, \tag{4.47b}$$

$$G_3^* = +\frac{\sqrt{2}}{8}\sinh^{N+1}\alpha\sum_{s=0}^{\infty}\{s(s+1)(2s+1)({}_aK_s)^R a_s + s(s+1)({}_bK_s)^R b_s + ({}_dK_s)^R d_s$$
$$+s(s-1)(s+1)(s+2)({}_fK_s)^R f_s\}. \tag{4.48b}$$

The values of ${}_cK_s$, ${}_eK_s$ and ${}_gK_s$ in (4.47a) and (4.48a) are polynomials in s like those shown in (4.26) but of degrees $N-2$ with $L, a_0, a_1, \dots a_{N-2}$ having the values listed in Table 9. Also in (4.47b) and (4.48b), ${}_aK_s$, ${}_bK_s$, ${}_dK_s$, and ${}_fK_s$ are polynomials in s like those shown in (4.26) with ${}_aK_s$ and ${}_fK_s$ of degree $N-3$ and ${}_bK_s$ and ${}_dK_s$ of degree $N-1$, the coefficients $L, a_0, a_1, \dots$ being listed in Table 10.

The values of ${}^Tc_s$, ${}^Te_s$, and ${}^Fg_s$ appearing in (4.47a) and ${}^Ta_s$, ${}^Tb_s$ ${}^Td_s$ and ${}^Tf_s$ appearing in (4.47b) are quantities appearing in the value of ${}^Su_{Ti2}$ determined by O'Neill [4] (see also O'Neill [18]) from which it is observed that ${}^Ta_s$ is determined by the infinite set of equations

$$[(2s-1)k_{s-1} - (2s-3)k_s]\left[\frac{(s-1)^T a_{s-1}}{(2s-1)} - \frac{s^T a_s}{(2s+1)}\right]$$
$$-\,[(2s+5)k_s - (2s+3)k_{s+1}]\left[\frac{(s+1)^T a_s}{(2s+1)} - \frac{(s+2)^T a_{s+1}}{(2s+3)}\right]$$
$$= \sqrt{2}\left[2\coth\left(s+\frac{1}{2}\right)\alpha - \coth\left(s-\frac{1}{2}\right)\alpha - \coth\left(s+\frac{3}{2}\right)\alpha\right], \quad (s \geq 1) \tag{4.49}$$

for ${}^Ta_1, {}^Ta_2, {}^Ta_3 \dots$ in which

$$k_s = \left(s+\frac{1}{2}\right)\coth\left(s+\frac{1}{2}\right)\alpha - \coth\alpha, \quad (s \geq 0). \tag{4.50}$$

${}^Tb_s$, ${}^Tc_s$, ${}^Td_s$, ${}^Te_s$, ${}^Tf_s$, and ${}^Tg_s$, are then given in terms of ${}^Ta_s$, by

$${}^Tb_s = (s-1)^T a_{s-1} - (2s+1)^T a_s + (s+2)^T a_{s+1}, \quad (s \geq 1), \tag{4.51}$$

Table 9. Values of the coefficients $L, a_0, a_1 \ldots$ (see (4.26)) for ${}_cK_s, {}_eK_s$, and ${}_gK_s$ appearing in (4.47a) and (4.48a) for the flow (4.45a,b) for N even ($N \leq 10$.) In the second column c, e, and g indicate whether the values are for ${}_cK_s, {}_e K_s$ or ${}_gK_s$.

N	c, e, g	L	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
2	c	$-\frac{4}{3}$	1								
	e	$-\frac{24}{5}$	1								
	g	0									
4	c	$-\frac{1276}{945}$	3	1	1						
	e	$-\frac{512}{189}$	3	2	2						
	g	$+\frac{512}{315}$	2	3	1						
6	c	$-\frac{7004}{45045}$	45	26	28	4	2				
	e	$-\frac{536}{715}$	15	16	18	4	2				
	g	$+\frac{1072}{3003}$	18	31	19	8	2				
8	c	$-\frac{7636}{240975}$	315	243	277	69	37	3	1		
	e	$-\frac{192}{80325}$	5985	8094	9880	3648	2014	228	76		
	g	$+\frac{1216}{11475}$	90	171	137	79	27	5	1		
10	c	$-\frac{56956}{62214075}$	14175	13212	15768	5252	2978	428	152	8	2
	e	$-\frac{1384}{226233}$	2835	4464	5796	2768	1646	320	116	8	2
	g	$-\frac{5536}{1382535}$	3150	6453	6073	4083	1683	477	117	12	2

Table 10. Values of the coefficients $L, a_0, a_1 \ldots$ (see (4.26)) for ${}_aK_s, {}_b K_s, {}_dK_s$ and ${}_fK_s$ appearing in (4.47 b) and (4.48 b) for the flow (4.45a,b) for N odd ($N \leq 10$.) In the second column a, b, d and f indicate whether the values are for ${}_aK_s, {}_b K_s, {}_d K_s$, or ${}_fK_s$.

N	$a, b, d,$ or f	L	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
3	a	$+\frac{48}{35}$	1								
	b	$-\frac{128}{35}$	1	1	1						
	d	$-\frac{384}{35}$	1	2	2						
	f	$+\frac{128}{35}$	1								
5	a	$-\frac{736}{99}$	3	1	1						
	b	$-\frac{224}{99}$	9	14	16	4	2				
	d	$-\frac{1120}{99}$	3	8	10	4	2				
	f	$+\frac{448}{99}$	3	2	2						
7	a	$-\frac{912}{585}$	45	26	28	4	2				
	b	$-\frac{128}{117}$	45	97	109	45	25	3	1		
	d	$-\frac{896}{585}$	45	138	196	120	70	12	4		
	f	$+\frac{384}{117}$	9	10	11	2	1				
9	a	$-\frac{15296}{33915}$	315	243	277	69	37	3	1		
	b	$-\frac{832}{14535}$	1575	3492	4688	2492	1498	308	112	8	2
	d	$-\frac{832}{2261}$	315	1056	1636	1232	798	224	84	8	2
	f	$+\frac{3328}{14535}$	225	324	383	120	65	6	2		

$$ {}^{T}c_s = -2k_s\left[\frac{(s-1)^{T}a_{s-1}}{(2s-1} - {}^{T}a_s + \frac{(s+2)^{T}a_{s+1}}{(2s+3)}\right], \quad (s \geq 1), \tag{4.52} $$

$$ {}^{T}d_s = -\frac{1}{2}(s-1)s^{T}a_{s-1} + \frac{1}{2}(s+1)(s+2)^{T}a_{s+1}, \quad (s \geq 0), \tag{4.53} $$

$$ {}^{T}e_s = \frac{2\sqrt{2}e^{-(s+\frac{1}{2})\alpha}}{\sinh(s+\frac{1}{2})\alpha} + k_s\left[\frac{(s-1)s^{T}a_{s-1}}{(2s-1)} - \frac{(s+1)(s+2)^{T}a_{s+1}}{(2s+3)}\right], \quad (s \geq 0), \tag{4.54} $$

$$ {}^{T}f_S = \frac{1}{2}({}^{T}a_{s-1} - {}^{T}a_{s+1}), \quad (s \geq 2), \tag{4.55} $$

$$ {}^{T}g_s = -k_s\left[\frac{{}^{T}a_{s-1}}{(2s-1)} - \frac{{}^{T}a_{s+1}}{(2s+1)}\right], \quad (s \geq 2). \tag{4.56} $$

The values of ${}^{T}a_0$, ${}^{T}b_0$, ${}^{T}c_0$, ${}^{T}f_0$, ${}^{T}f_1$, ${}^{T}g_0$, and ${}^{T}g_1$ are not defined by these equations, but they are not needed anyway since they do not affect the values of F_2^* given by (4.47a,b).

Likewise, in (4.48a,b), the quantities ${}^{R}a_s$, ${}^{R}b_s$, ${}^{R}c_s$, ${}^{R}d_s$, ${}^{R}e_s$, ${}^{R}f_s$ and ${}^{R}g_s$, are those appearing in the value of ${}^{S}u_{Ri3}$ determined by Dean & O'Neill [8], with ${}^{R}a_s$ being determined by the infinite set of equations

$$ \begin{aligned} &[(2s-1)k_{s-1} - (2s-3)k_s]\left[\frac{(s-1)^{R}a_{s-1}}{(2s-1)} - \frac{s^{R}a_s}{(2s+1)}\right] \\ &- [(2s+5)k_s - (2s+3)k_{s+1}]\left[\frac{(s+1)^{R}a_s}{(2s+1)} - \frac{(s+2)^{R}a_{s+1}}{(2s+3)}\right] \\ &= -\frac{\sqrt{2}e^{-(s+\frac{1}{2})\alpha}}{(2s+1)\sinh\alpha}\left[(2s+1)^2\left(\frac{e^{\alpha}}{(2s-1)} + \frac{e^{-\alpha}}{(2s+3)}\right)\operatorname{cosech}(s+\frac{1}{2})\alpha\right. \\ &\left.-(2s-1)\operatorname{cosech}(s-\frac{1}{2})\alpha - (2s+3)\operatorname{cosech}(s+\frac{3}{2})\alpha\right], \quad (s \geq 1) \end{aligned} \tag{4.57} $$

for ${}^{R}a_1, {}^{R}a_2, {}^{R}a_3 \ldots$. Then ${}^{R}b_s$, ${}^{R}d_s$ and ${}^{R}f_s$ are given in terms of ${}^{R}a_s$ by the same equations as for ${}^{T}b_s$, ${}^{T}d_s$, and ${}^{T}f_s$ [i.e. by Eqs. (4.51), (4.53) and (4.55) with ${}^{T}a_s$, ${}^{T}b_s$, ${}^{T}d_s$ and ${}^{T}f_s$ replaced by ${}^{R}a_s$, ${}^{R}b_s$, ${}^{R}d_s$ and ${}^{R}f_s$]. The remaining quantities ${}^{R}c_s$, ${}^{R}e_s$ and ${}^{R}g_s$ are then

$$ \begin{aligned} {}^{R}c_s = {} & 4\lambda_s \operatorname{cosech}\alpha \operatorname{cosech}(s+\frac{1}{2})\alpha \\ & -2k_s\left[\frac{(s-1)^{R}a_{s-1}}{(2s-1)} - {}^{R}a_s + \frac{(s+2)^{R}a_{s+1}}{(2s+3)}\right], \quad (s \geq 1), \end{aligned} \tag{4.58} $$

$$ \begin{aligned} {}^{R}e_s = {} & \left[\sqrt{2}(2s+1)e^{-(s+\frac{1}{2})\alpha} - \lambda_s \operatorname{cosech}\alpha\right]\operatorname{cosech}(s+\frac{1}{2})\alpha \\ & +k_s\left[\frac{(s-1)s^{R}a_{s-1}}{(2s-1)} - \frac{(s+1)(s+2)^{R}a_{s+1}}{(2s+3)}\right], \quad (s \geq 0), \end{aligned} \tag{4.59} $$

$$ {}^{R}g_s = -4\lambda_s \operatorname{cosech}\alpha \operatorname{cosech}(s+\frac{1}{2})\alpha - k_s\left[\frac{{}^{R}a_{s-1}}{(2s-1)} - \frac{{}^{R}a_{s+1}}{(2s+3)}\right], \quad (s \geq 2), \tag{4.60} $$

where

$$\lambda_s = -\frac{1}{\sqrt{2}}\left[\frac{e^{-(s-\frac{1}{2})\alpha}}{(2s-1)} - \frac{e^{-(s+\frac{3}{2})\alpha}}{(2s+3)}\right], \quad (s \geq 0). \tag{4.61}$$

Using the results (4.49)–(4.61) and the Tables 9 and 10, the dimensionless force F_2^* and moment of force G_3^* on the sphere were calculated numerically as a function of h/a from (4.47a) and (4.48a) for flows of type (4.45a) for N even and from (4.47b) and (4.48b) for flows of type (4.45b) for N odd. Again, to illustrate the results, it is more convenient to define a new dimensionless force $\overline{F}_2^*$ as

$$\overline{F}_2^* = \frac{F_2}{6\pi\mu a U_{C2}}, \tag{4.62}$$

so that $\overline{F}_2^* \to 1$ as $h^* \to \infty$, and a new dimensionless moment of force

$$\overline{G}_3^* = \frac{G_3}{4\pi\mu a^3 \omega_{C3}}, \tag{4.63}$$

so that $\overline{G}_3^* \to 1$ as $h^* \to \infty$. $\overline{F}_2^*$ and $\overline{G}_3^*$ are then respectively related to F_2^* and G_3^* (given by (4.47) and (4.48)) by

$$\overline{F}_2^* = -\frac{F_2^*}{2N\cosh^{N-1}\alpha} \tag{4.64}$$

and

$$\overline{G}_3^* = -\frac{G_3^*}{2(N-1)\cosh^{N-2}\alpha} \tag{4.65}$$

for flows of type (4.39a) for N even, and by

$$\overline{F}_2^* = -\frac{F_2^*}{2N(N-1)\cosh^{N-1}\alpha} \tag{4.66}$$

and

$$\overline{G}_3^* = -\frac{G_3^*}{2N(N-1)\cosh^{N-2}\alpha} \tag{4.67}$$

for flows of type (4.39b) for N odd.

The results for the dimensionless force $\overline{F}_2^*$ and moment of force $\overline{G}_3^*$ acting on the sphere as a function of h^* have been plotted respectively in Figs. 8a and 8b for all flows (4.45a,b) with $N \leq 10$.

Finally, it is seen directly from O'Neill [4] that for the sphere translating (but not rotating) with velocity $\hat{U}$ in the 2-direction in a quiescent fluid bounded by a solid surface, the dimensionless force $\overline{F}_2^*$ on the sphere in the 2-direction (made dimensionless by $6\pi\mu a\hat{U}$ so that $\overline{F}_2^* \to -1$ as $h^* \to \infty$) is

$$\overline{F}_2^* = -\frac{\sqrt{2}}{6}\sinh^2\alpha \sum_{s=0}^{\infty}\{s(s+1)^T c_s + {}^T e_s\}, \tag{4.68}$$

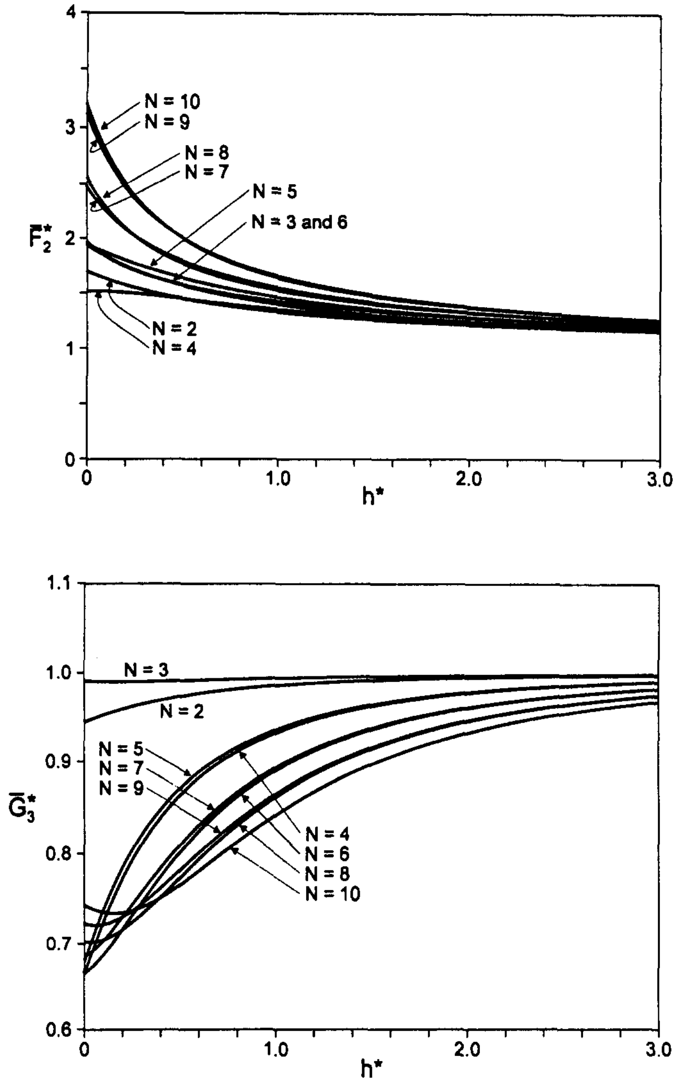


Fig. 8. Force $\overline{F}_2^*$ shown in (a) and moment of force $\overline{G}_3^*$ shown in (b) as a function of h^* for 'antisymmetric' flows bounded by a solid surface of $r_1 = 0$. The flows are given by (4.45a) for N even and by (4.45b) for N odd. Note: In Figure 8a, values for $N = 6$ are greater than those for $N = 3$ for $h^* > 0.5$.

whilst the dimensionless moment of force $\overline{G}_3^*$ on the sphere about its centre in the 3-direction (made dimensionless by $8\pi\mu a^2 \hat{U}$ so that $\overline{G}_3^* \to 0$ as $h* \to \infty$) is

$$\begin{aligned}\overline{G}_3^* = \frac{\sinh^2\alpha}{12\sqrt{2}} \sum_{s=0}^{\infty} \Big\{ &\left[2+e^{(2s+1)\alpha}\right]\left[s(s+1)(2(^T a_s)+\coth\alpha(^T c_s)) - (2s+1-\coth\alpha)^T e_s\right] \\ &+\left[2-e^{-(2s+1)\alpha}\right]\left[s(s+1)\coth\alpha(^T b_s) - (2s+1-\coth\alpha)^T d_s\right]\Big\}. \qquad (4.69)\end{aligned}$$

Also, from Dean & O'Neill [8], we see that for the sphere rotating about its centre (but not translating) with angular velocity $\hat{\Omega}$ in the 3-direction, the dimensionless force $\overline{F}_2^*$ on the sphere in the 2-direction (made dimensionless by $6\pi\mu a^2\hat{\Omega}$ so that $\overline{F}_2^* \to 0$ as $h^* \to \infty$) is

$$\overline{F}_2^* = -\frac{\sqrt{2}}{6}\sinh^2\alpha\sum_{s=0}^{\infty}\{s(s+1)^R c_s + {}^R e_s\}, \tag{4.70}$$

whilst the dimensionless moment of force $\overline{G}_3^*$ on the sphere about its centre in the 3-direction (made dimensionless by $8\pi\mu a^3\hat{\Omega}$ so that $\overline{G}_3^* \to -1$ as $h^* \to \infty$) is

$$\begin{aligned}\overline{G}_3^* \;=\; & -\frac{1}{3}+\frac{\sinh^3\alpha}{12\sqrt{2}}\sum_{s=0}^{\infty}\left[2+e^{-(2s+1)\alpha}\right]\left[s(s+1)(2({}^R a_s)+\coth\alpha({}^R c_s))-(2s+1-\coth\alpha)^R e_s\right] \\ & +\left[2-e^{-(2s+1)\alpha}\right]\left[(s+1)(\coth\alpha)^R b_s-(2s+1-\coth\alpha)^R d_s\right].\end{aligned} \tag{4.71}$$

5. Exact motion of a solid sphere

We consider now the solid sphere freely moving with zero external force and moment of force (about its centre) acting on it in a fluid occupying the region $r_1 > 0$ and bounded by a undeformable free surface at $r_1 = 0$. It is assumed that the fluid undergoes a two dimensional undisturbed flow **U** which is a linear combination of the flows (4.3a,b) and (4.12a,b) considered in sections 4.1 and 4.2 with $N \leq 10$. Its streamfunction ψ is thus of the form

$$\begin{aligned}\psi \;=\; & \sum_{\substack{N=2\\ N\,\text{even}}}^{10} A_N\rho^N\sin N\theta+\sum_{\substack{N=4\\ N\,\text{even}}}^{10} B_N\rho^N\sin(N-2)\theta+\sum_{\substack{N=3\\ N\,\text{odd}}}^{9} C_N\rho^N\cos N\theta \\ & +\sum_{\substack{N=3\\ N\,\text{odd}}}^{9} D_N\rho^N\cos(N-2)\theta,\end{aligned} \tag{5.1}$$

where A_N, B_N, C_N and D_N are constants. In terms of the Cartesian coordinates r_1, r_2 this streamfunction may be written as the double sum

$$\psi = \sum_{m}\sum_{n}{}_{(m+n\,\leq\,10)}\; a_{mn}r_1^m r_2^n \tag{5.2}$$

in which, of course, the a_{mn} are not independent.

If at any instant of time t_0 the sphere centre is at position (r_1^*, r_2^*) we define a new set of coordinates $\overline{r}_1, \overline{r}_2$ with origin at O* as shown in Fig. 9 so that

$$\overline{r}_1 = r_1, \quad \overline{r}_2 = r_2 - r_2^*. \tag{5.3}$$

Substituting this in (5.2) we see that in terms $\overline{r}_1, \overline{r}_2$

$$\psi = \sum_{m}\sum_{n}{}_{(m+n\,\leq\,10)}\; b_{mn}\overline{r}_1^m \overline{r}_2^n. \tag{5.4}$$

Since this flow must satisfy the creeping flow equations with the free surface boundary conditions on $r_1 = 0$, this value of ψ must be a linear combination of flows (4.3a,b) and (4.12a,b) with origin at O* [i.e. with ρ, θ replaced by $\overline{\rho}, \overline{\theta}$, the polar coordinates with origin at O* (see Fig. 9)]. Since the problem of the calculation of the force and moment of force on

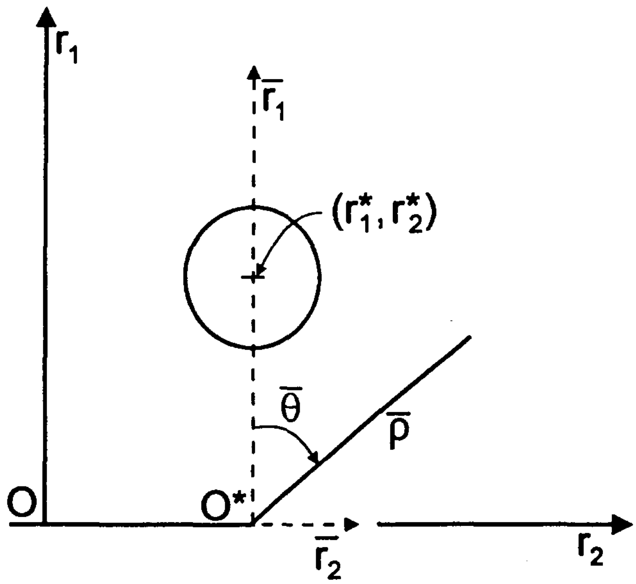


Fig. 9. Definition of $\overline{r_1}, \overline{r_2}$ coordinates.

the sphere at rest in this flow is linear, the results of sections 4.1 and 4.2 with $h = r_1^*$ may be used to calculate the force $\mathbf{F} = (F_1, F_2, 0)$ and moment of force $\mathbf{G} = (0, 0, G_3)$ on the sphere at rest in the flow. Then by further use of linearity (see for example Happel & Brenner [3] §8.5), we are assured that at its position at time t_0, the sphere, when freely allowed to move, has a velocity $\mathbf{V} = (V_1, V_2, 0)$ and angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega_3)$ determined by the total hydrodynamic force and moment of force (about its centre) being zero, i.e. by

$$F_1 + \alpha_1 V_1 = 0, \tag{5.5a}$$

$$F_2 + \beta_{22} V_2 + \beta_{23} \Omega_3 = 0, \tag{5.5b}$$

$$G_3 + \beta_{32} V_2 + \beta_{33} \Omega_3 = 0, \tag{5.5c}$$

where the resistance coefficients $\alpha_1, \beta_{22}, \beta_{23} = \beta_{32}$ and β_{33} are calculated using (4.11) and (4.33)–(4.36). The velocity components of the sphere are thus obtained [V_1] from (5.5a) and $V2$ from (5.5b) and (5.5c)] for the known sphere position (r_1^*, r_2^*) at the time t_0. From this, one may calculate the position of the sphere at a slightly later time $t_0 + \Delta t$. By repeating this procedure one may calculate numerically the orbit of the sphere centre for a freely moving sphere in the given flow (5.1).

In a similar manner, one may calculate also the orbit of a sphere freely moving in a given undisturbed two dimensional flow in the region $r_1 > 0$ bounded by a solid wall at $r_1 = 0$. In such a case the undisturbed flow would have a streamfunction which would be a linear combination of those given by (4.37 a,b) and (4.45 a,b), and the calculation would require the use of the results of the sections 4.3 and 4.4.

5.1. AN EXAMPLE

As an example of the calculation just described of the motion of a solid sphere, we consider such a sphere freely moving (with zero force and moment of force about its centre) in a fluid

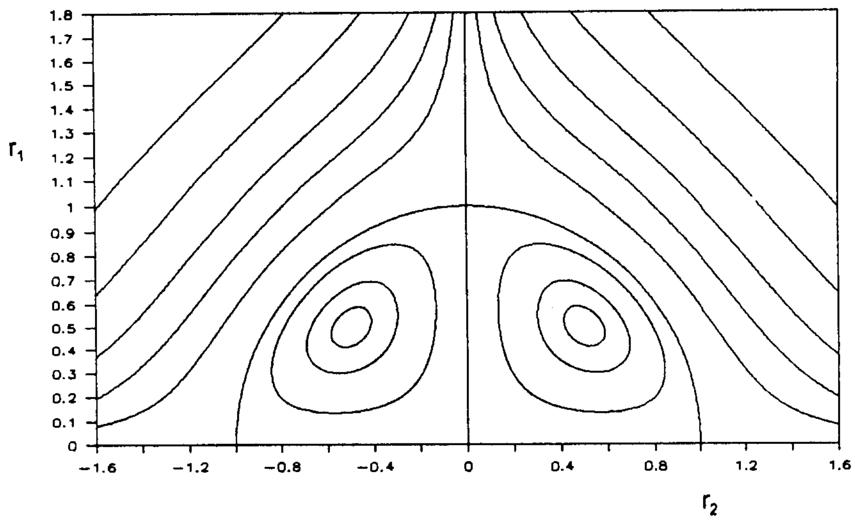


Fig. 10. Streamlines of vortex flow with undeformable free surface at $r_1 = 0$. For streamfunction (5.7) the flow is anticlockwise in the right hand vortex (in $r_2 > 0$), whilst for the streamfunction (6.2) it is clockwise.

occupying the region $r_1 > 0$ which is undergoing a prescribed two dimensional undisturbed velocity field $\mathbf{U}$ and is bounded by an undeformable free surface at $r_1 = 0$. For this prescribed undisturbed flow we choose one which is a particular linear combination of a flow of type (4.3a) with $N = 2$ and a flow of type (4.3b) with $N = 4$, namely we take $\mathbf{U}$ to be a flow with streamfunction $\psi\ (r_1, r_2)$ given by

$$\psi = \frac{1}{2}\rho^2(\rho^2 - 1)\sin 2\theta \tag{5.6}$$

in terms of plane polar coordinates (ρ, θ) with an origin O on the boundary $r_1 = 0$ (see Fig. 2). This expression for ψ can be considered as being dimensionless with ψ and ρ having been made non-dimensional by a characteristic velocity U^* and length L^*. In terms of Cartesian coordinates r_1, r_2 (with origin at O) the stream function ψ is

$$\psi = r_1 r_2^3 + r_1^3 r_2 - r_1 r_2, \tag{5.7}$$

giving the undisturbed velocity $\mathbf{U} = (U_1, U_2)$ as

$$U_1 = 3r_1 r_2^2 + r_1^3 - r_1, \tag{5.8}$$

$$U_2 = -r_2^3 - 3r_1^2 r_2 + r_2.$$

This flow, shown in Fig. 10, is symmetric about $r_2 = 0$ and consists of a pair of counter-rotating vortices which are bounded by the boundary, the symmetry axis $r_2 = 0$ and the semicircle $\rho = 1$ of unit radius (or of dimensional radius L^*). The centres of each of the vortices where $\mathbf{U} = \mathbf{0}$ are at $\left(\frac{1}{2}, \pm\frac{1}{2}\right)$.

In the manner previously described (see §5), the orbits of the centre of a freely moving solid sphere in this flow have been calculated numerically for various values of a/L^*, the ratio of sphere radius to vortex size. These results (for $a/L^* = 0.2$, 0.25, 0.3 and 0.35) are shown in Fig. 11a–11d. It is observed that for small a/L^* the sphere centre, whilst almost following the undisturbed flow, slowly spirals into a point very close to the vortex centre (at position $\left(\frac{1}{2}, \pm\frac{1}{2}\right)$ in dimensionless variables). As a/L^* is increased, the spiral motion is more rapid (with the sphere centre moving inwards a greater distance for each revolution around the vortex) and spirals into a point which moves away from the vortex centre somewhat. However, if a/L^* is increased beyond a value of about 0.4, there is no spiral motion with the sphere centre moving on an open orbit. In fact the sphere is then so large compared with the vortex that in its general behaviour it no longer recognizes the existence of the vortex.

6. Discussion of results

In §§2 – 4 it was shown how one may calculate at zero Reynolds number the force and moment of force on a solid sphere placed at rest in a flowing viscous fluid (with velocity **U** and streamfunction ψ) occupying a semi-infinite region bounded at $r_1 = 0$ by either an undeformable free surface (with undisturbed flow which is a linear combination of (4.3a), (4.3b), (4.12a) and (4.12b) with $N \leq 10$) or a solid surface (with undisturbed flow which is a linear combination of (4.37a), (4.37b), (4.45a) and (4.45b) with $N \leq 10$).

These results were then used in §5 to obtain the orbit of a freely moving sphere in a fluid bounded by an undeformable free surface at $r_1 = 0$ in which the undisturbed velocity field **U** is chosen to be that due to the (dimensionless) streamfunction

$$\psi = r_1 r_2^2 + r_1^3 r_2 - r_1 r_2. \tag{6.1}$$

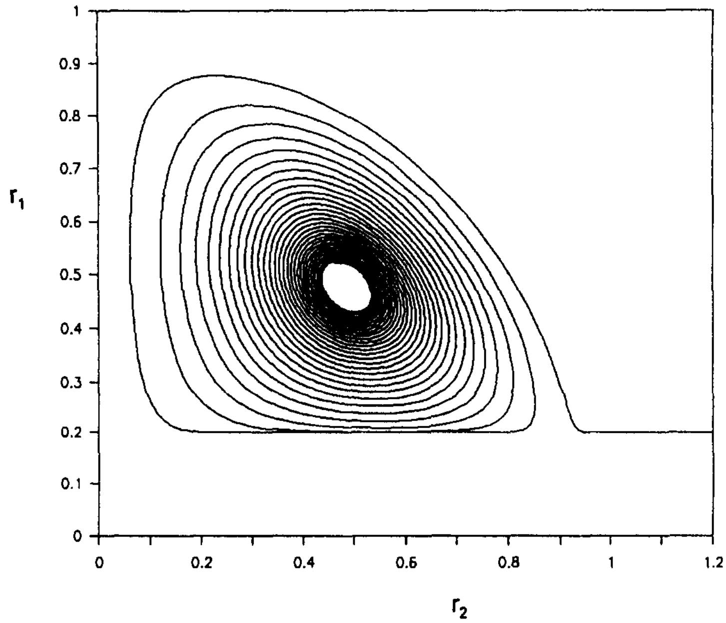
This was done for several values of a/L^*, the ratio of the sphere radius a to vortex size L^*. It was observed that whereas the undisturbed flow has a single bounded anticlockwise vortex (in the region $r_1 > 0$) in which all streamlines are closed (see Fig. 10), the motion of the sphere centre is such that after entering the vortex it spirals around the vortex, moving across streamlines until it ends up at a point Q close to (but not exactly at) the vortex centre P where the undisturbed flow velocity is zero (see Figs. 11a–11d). This migration of the sphere across streamlines is very slow when a/L^* is small but the process speeds up as a/L^* is increased. No spiral motion occurs with a/L^* larger than about 0.4.

Since the problem of finding the sphere velocity is linear, it follows that if the undisturbed flow velocity **U** is everywhere reversed so that it represents a bounded clockwise vortex (in the region $r_2 > 0$) with

$$\psi = -r_1 r_2^3 - r_1^3 r_2 + r_1 r_2 \tag{6.2}$$

then the sphere motion is exactly reversed and hence moves along the paths shown in Figs. 11a–11d but in the reverse direction. Under such a situation a sphere placed in the vortex will spiral around the vortex, moving outwards away from the vortex centre until it eventually leaves the vortex by moving in the r_2 - direction whilst almost being in contact with the free surface $r_1 = 0$.

If instead of a single spherical particle in the undisturbed flow, one has a dilute suspension of such spheres (with a/L^* small but non-zero) with a concentration so low that hydrodynamic sphere-sphere interactions may be ignored, then for the flow (6.1) the particles would concentrate in the vortex, with the concentrate becoming larger and larger at the point Q as time proceeds. In fact at steady state the concentration at Q would be infinite. In the real situation however, sphere-sphere interactions must become important in the neighbourhood of Q at some stage. For the reverse flow (6.2), particles initially in the vortex would slowly spiral out of the vortex, leaving a region (which would be approximately the vortex region



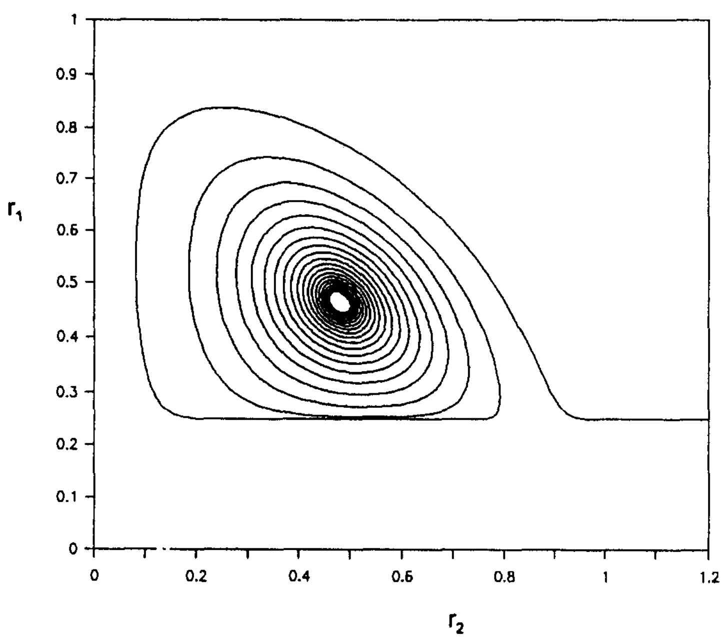


*Fig. 11*a–b. Path of sphere centre in the vortex flow with the free surface present at $r_1 = 0$. The values of α/L^* are respectively 0.2, 0.25, 0.3 and 0.35 in the figures 11a, b, c and d. For the undisturbed flow (5.7), the sphere moves into the vortex and spirals inwards in an anticlockwise manner. For the undisturbed flow (6.2), this sphere motion is reversed.

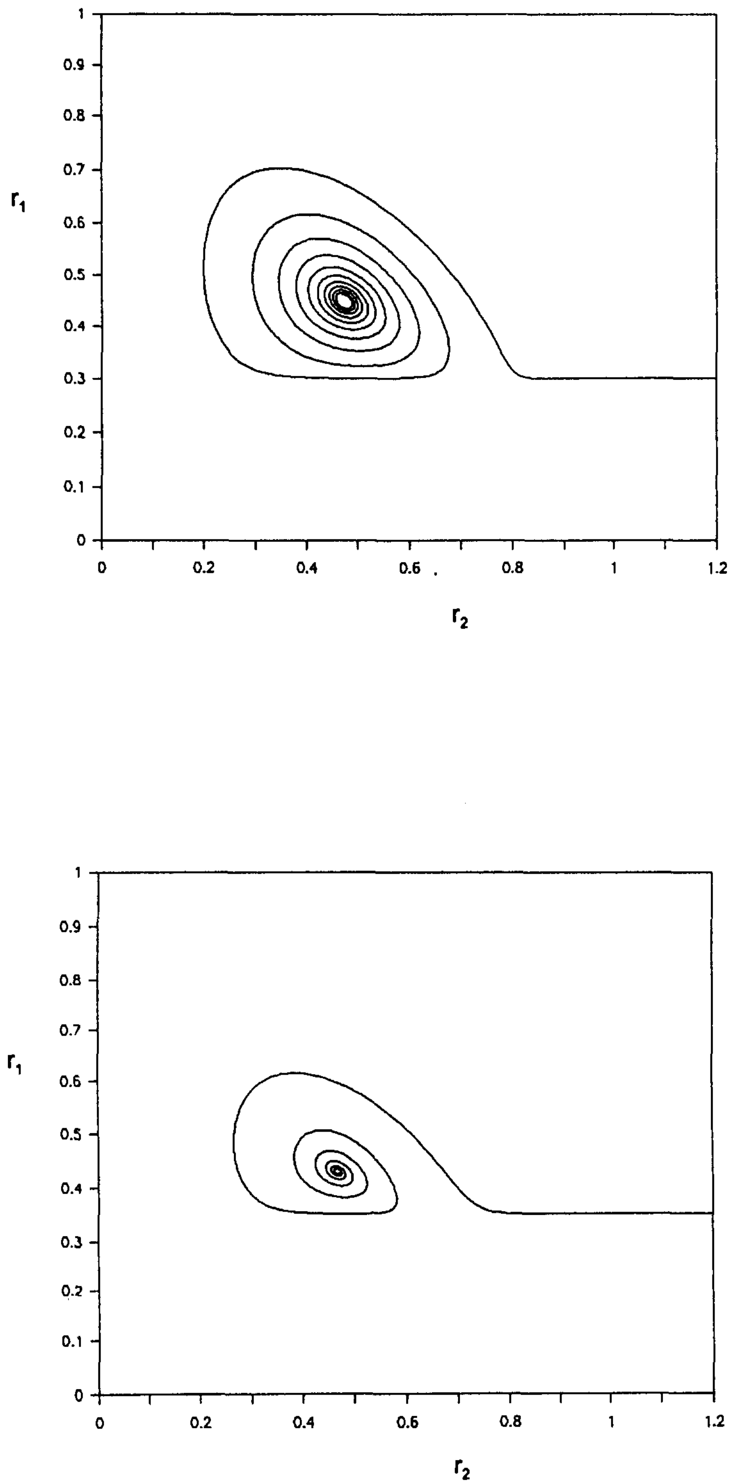


*Fig. 11*c–d. (See caption for Fig. 11a–b.)

itself for small a/L^*) which would become devoid of spheres, so that for all future times the concentration would be zero there.

It should be noted that in a dilute suspension of spheres (in which sphere-sphere interactions are neglected) if the sphere velocity is $\mathbf{V}$ at any position in the undisturbed flow $\mathbf{U}$ then the sphere concentration c satisfies the steady state the continuity equation

$$\mathbf{V}\cdot\nabla(\log c) = -\nabla\cdot\mathbf{V}. \tag{6.3}$$

In the limit of zero particle size (i.e. as $a/L^* \rightarrow 0$) the particle velocity $\mathbf{V}$ will approach $\mathbf{U}$, the undisturbed fluid velocity, and since $\nabla\cdot\mathbf{U} = 0$ it follows that the particle concentration c given at steady state by (6.3) will now satisfy

$$\mathbf{U}\cdot\nabla c = 0, \tag{6.4}$$

so that there is a steady state solution in which c is constant everywhere, there being no tendency for the particles to concentrate in any particular region (or to leave any particular region) of the flow.

For non-zero a/L^*, if one has an undisturbed fluid flow $\mathbf{U}$ in an *unbounded* region (with no boundaries present) then the velocity $\mathbf{V}$ of the centre of a freely moving sphere is determined by Faxén's laws (see (1.1)) by equating the dimensionless force $\overline{\mathbf{F}}^*$ on the sphere to zero to give

$$\mathbf{0} = \overline{\mathbf{F}}^* = \mathbf{U}|_C - \mathbf{V} + \frac{1}{6}(a/L^*)^2(\nabla^2\mathbf{U})|_C \tag{6.5}$$

or

$$\mathbf{V} = \mathbf{U}|_C + \frac{1}{6}(a/L^*)^2(\nabla^2\mathbf{U})|_C, \tag{6.6}$$

where $|_C$ indicates evaluation at the sphere centre. Regarding the sphere velocity $\mathbf{V}$ as a function of position we observe from (6.5) that

$$\nabla\cdot\mathbf{V} = 0, \tag{6.7}$$

so that the Eq. (6.3) for the sphere concentration c reduces to

$$\mathbf{V}\cdot\nabla c = 0. \tag{6.8}$$

Thus again there is a steady state solution for which the concentration c is a constant everywhere, there being no tendency for the particles to accumulate in any region (or to leave any region) of the flow. It may also be readily shown from (6.5) that the paths of individual sphere centres for the two dimensional undisturbed flow considered in §3 are lines given by

$$\psi + \frac{1}{6}(a/L^*)^2\nabla^2\psi = \text{constant} \tag{6.9}$$

for which the spiral motion towards or away from a point Q is impossible (since the stream-function ψ is not singular within the region occupied by the fluid). However note that whilst there is, for a suspension undergoing an unbounded flow, no tendency for the spheres to accumulate in (or to leave) any region of the flow, the motion of any particular sphere (given by (6.8)) is not the same as that of the undisturbed motion of the fluid (with streamlines ψ = constant) so that in general the sphere does move locally across streamlines.

Also, even for an undisturbed two-dimensional vortex flow of a fluid bounded at $r_1 = 0$ by an undeformable free surface (or by a solid wall), a freely moving solid sphere with non-zero a/L^* will not have the spiral orbits shown in Fig. 11 (or tend to concentrate at a point Q) if the vortex has fore-aft mirror symmetry in the r_2 direction, this following directly from the symmetry and the linearity of the problem.

Thus, at zero Reynolds number, for freely moving solid spherical particles to concentrate within a vortex (or to leave a vortex) as in §5, it is necessary that (i) the spherical particles have non-zero size and that (ii) there is a boundary such as a free surface or solid wall present. Also for the situation in which one has a two-dimensional undisturbed flow in the region $r_1 > 0$ bounded by a free surface or solid wall at $r_1 = 0$, the vortex should *not* possess mirror symmetry in the r_2 direction.

This predicted phenomenon, of the effect of (free surface or solid) boundaries on the motion of solid spherical particles in a prescribed undisturbed flow at zero Reynolds number causing the particles to move across streamlines and resulting in certain regions of the flow increasing (or decreasing) their concentration of particles, has been observed experimentally in a number of different situations. For example, Forgacs *et al.* [19] and Karnis *et al.* [20] experimentally examined what is known as the 'meniscus effect', in which particles suspended in a liquid flowing along a capillary tube behind an advancing meniscus tend to concentrate in a region immediately behind the meniscus. This effect is rather similar to that discussed above but with a more complicated system of boundaries present. Also it has been observed by Karino & Goldsmith [21] that when blood flows from a narrow into a wider radius capillary tube, the red blood cells move out of the vortex in the flow behind the tube expansion, leaving the vortex devoid of red blood cells. This may occur as a result of boundary effects similar to that discussed in this paper, but it is probable that the deformability of the red cells plays a role in this phenomenon (see Chaffey *et al.* [22]). Recently Gu [23] has reported observing experimentally the spiral motion of solid spherical particles into and out of an asymmetric vortex in a two dimensional flow bounded by a plane solid wall. This situation is essentially the same as that considered in §5 except that the planar boundary is solid rather than being an undeformable free surface.

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